Universität Münster

Paradoxical sets and the Axiom of Choice

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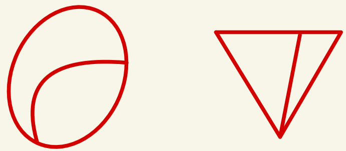
MM Mathematics Münster Cluster of Excellence

Set theory

- Model of ZF(C) = "universe" → There are several

Axiom of Choice



How does constructing models look like?

Set theory

How does constructing models look like?

$$(M\,,\ \mathbb{P},\ g \subseteq \mathbb{P}) \longrightarrow \boxed{\text{Forcing}} \rightsquigarrow M[g]$$

Context

Axiom of Choice

Paradoxical sets

Axiom of Choice

Axiom of Choice

$\Longleftrightarrow$

Zorn's lemma

$\Longleftrightarrow$

Well-ordering principle

$\Downarrow$

"$\mathbb{R}$ can be well-ordered"

Paradoxical sets

- $P \subseteq \mathbb{R}^n$ $\quad n = 1, 2, 3.$

P satisfies some counterintuitive property.

The existence of such a P involves Axiom of Choice

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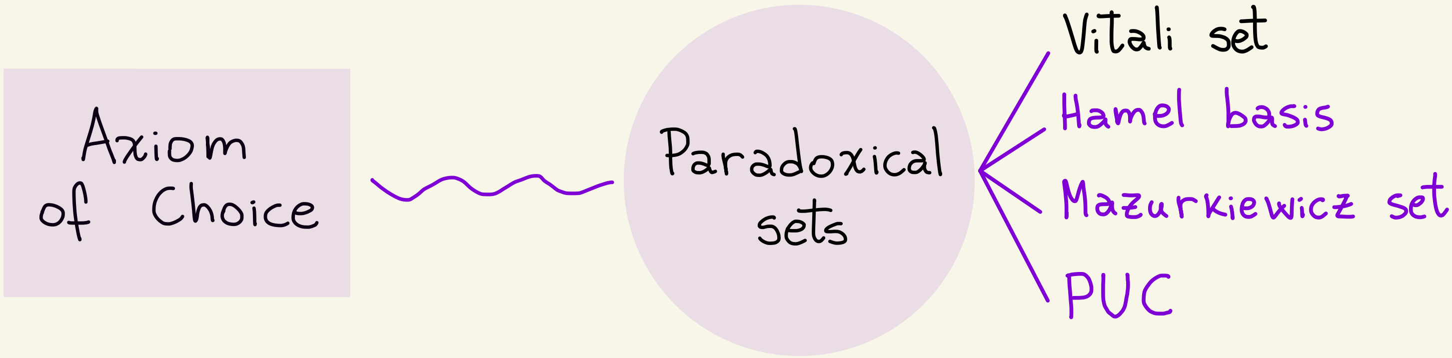
$\Downarrow$

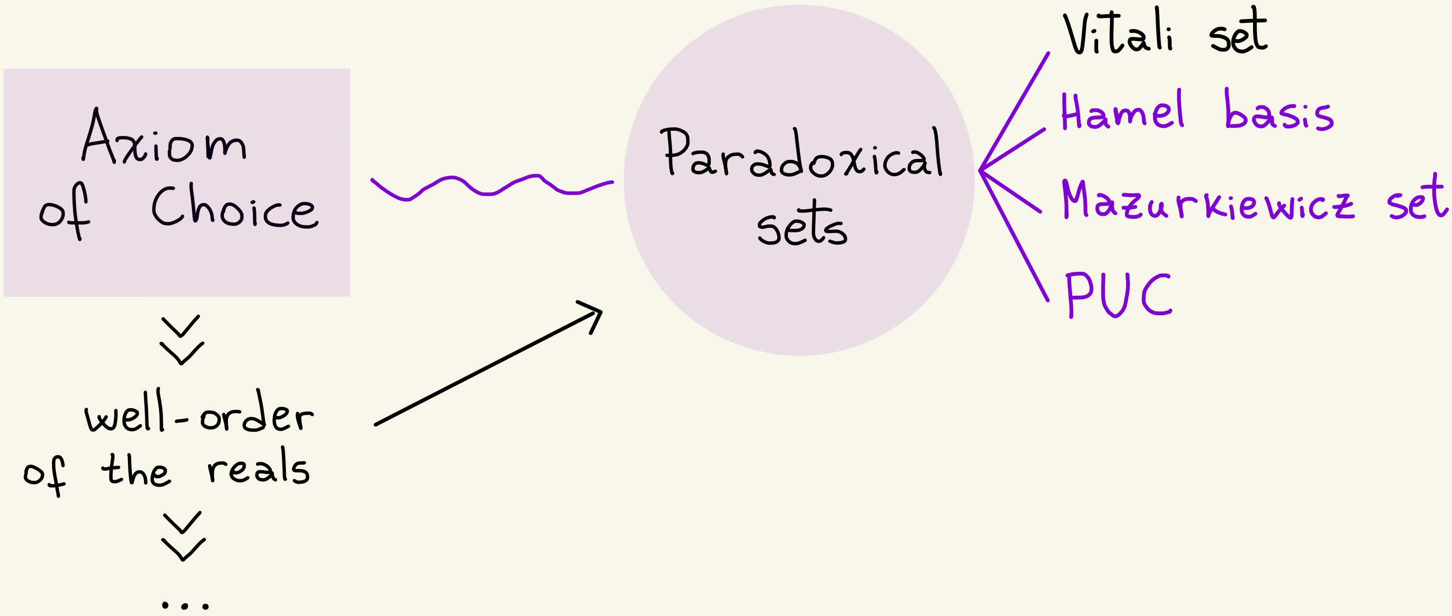
a well-order of the reals

allows induction "on the reals"

Context


Axiom of Choice
Paradoxical sets
Vitali set
Hamel basis
Mazurkiewicz set
PUC

Context



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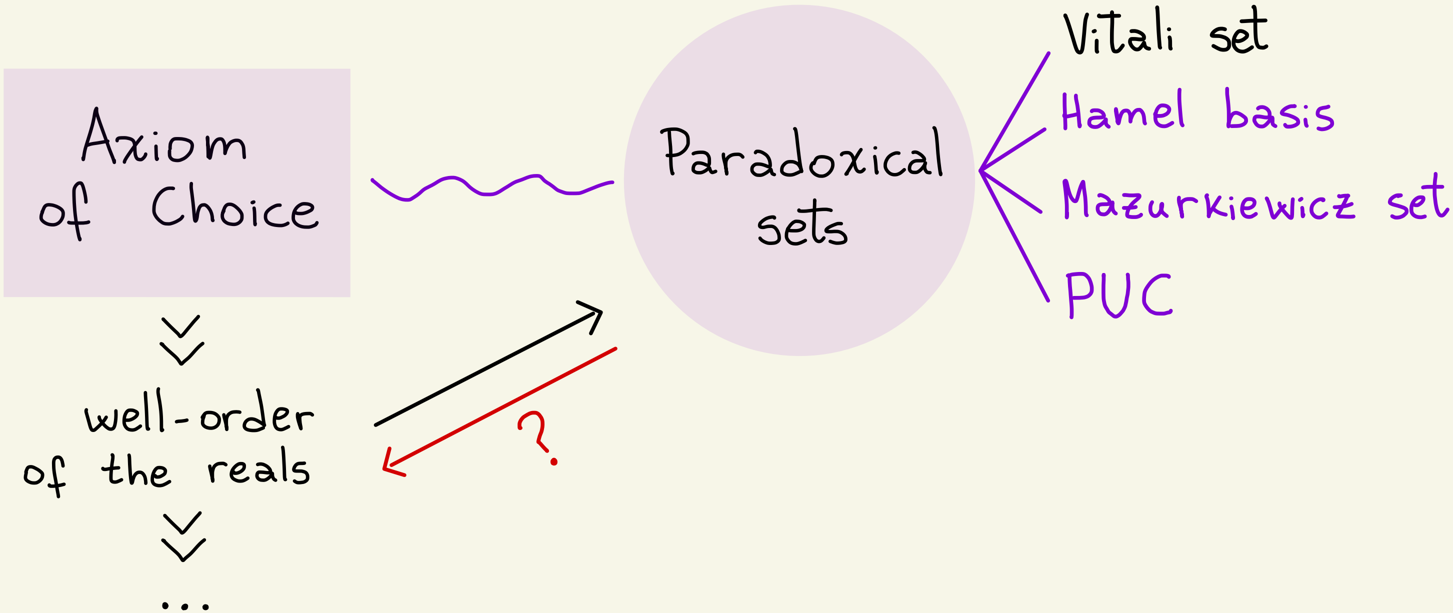
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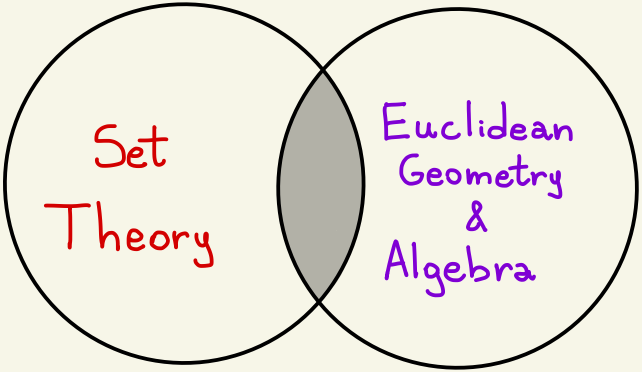
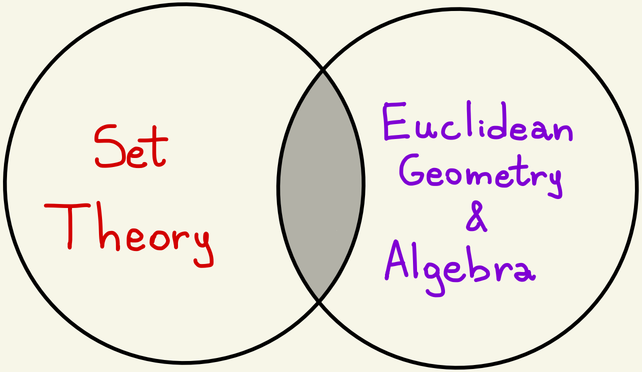
Contribution

Negative answer $\to$ Model of $ZF + \neg C + \exists P$



Contribution

Negative answer $\rightarrow$ Model of $ZF + \neg C + \exists P$



Strategy:
- Construct a model (of $ZF + \neg C$) which contains a nice structure of inner models satisfying AC.
- P will be a limit of the partial paradoxical sets.

Recent results

Models of $ZF + \neg C + \exists P$

- First Cohen model $L(A)$:

  $ZF + \neg AC_\omega + \ldots$ + Hamel basis [BSWY]

  + Mazurkiewicz set [BS]

  + Partition $\mathbb{R}^3$ in unit circles [F]

Recent results

Models of $ZF + \neg C + \exists P$

* Models of $ZF + DC + \neg WO(\mathbb{R}) +$

- Hamel basis [SY]
- Hamel basis + ... [B'CSWY]
- Mazurkiewicz set [BS]
- Partition of $\mathbb{R}^3$ in unit circles [F]

Recipe I

Dish: $L(\mathbb{R}, \bigcup h)^{V[g,h]} \models ZF + DC + \neg WO(\mathbb{R}) + \Psi(\mathcal{P})$

Ingredients:

- $V \models ZFC$.
- $\mathbb{Q} = \mathbb{C}(\omega_1)$, g $\mathbb{Q}$-generic filter over V.
- $\mathbb{P} \in V[g]$, h $\mathbb{P}$-generic filter over $V[g]$, $\mathcal{P} := \bigcup h$.

  $\hookrightarrow$ $\mathbb{P}$ adds a real partition according to Ψ.

Preparation:

- Prove $\mathbb{P}$ satisfies Extendability and Amalgamation.

Recipe II

Dish: $L(\mathbb{R}, \mathcal{U}h)^{V[g,h]} \models ZF + DC + \neg \mathsf{Ul}(\omega) + \Psi(\mathcal{P})$

Ingredients:

- $V \models ZFC$.
- $\mathbb{Q} = \mathbb{C}(\omega_1)$, g $\mathbb{Q}$-generic filter over V.
- $\mathbb{P} \in V[g]$, h $\mathbb{P}$-generic filter over $V[g]$, $\mathcal{P} := \mathcal{U}h$.
  - $\hookrightarrow$ $\mathbb{P}$ adds a real partition according to Ψ.

Preparation:

- Prove $\mathbb{P}$ satisfies Extendability and strong Amalgamation.

Recipe I

<u>Dish</u>: $L(\mathbb{R}, \bigcup h)^{V[g,h]} \models \mathrm{ZF} + \mathrm{DC} + \neg \mathrm{WO}(\mathbb{R}) + \Psi(\mathcal{P})$

<u>Ingredients</u>:

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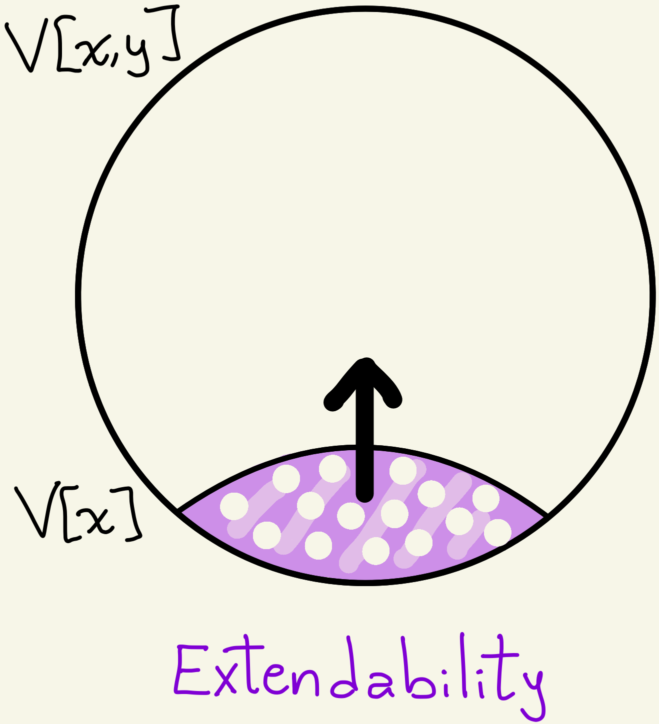
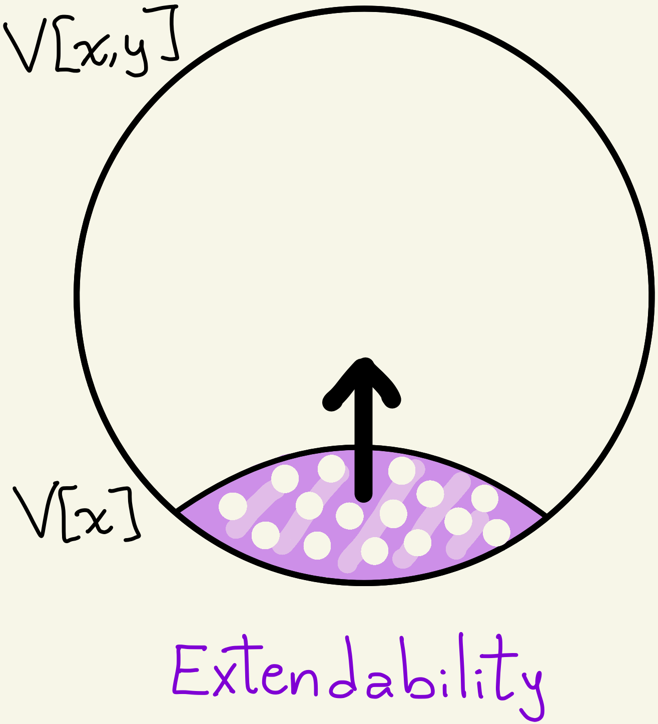
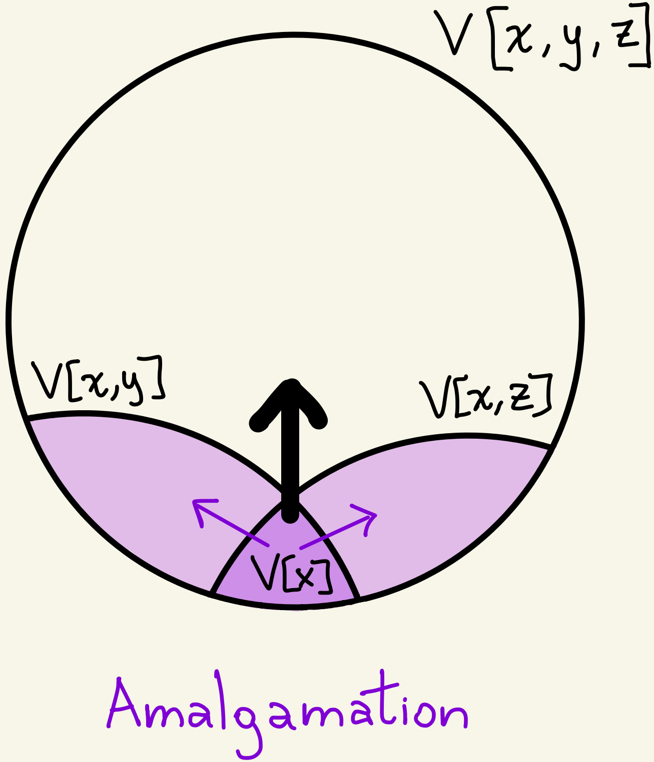
Recipe I

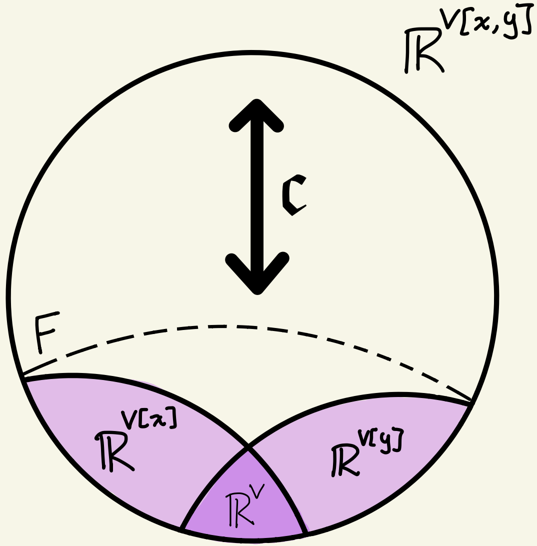


Recipe I





Algebra

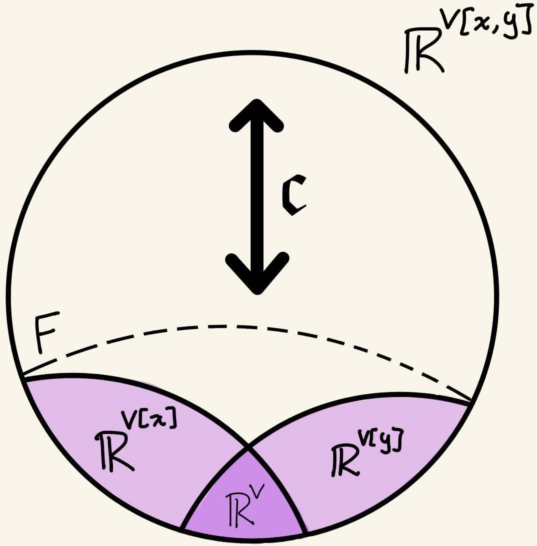


Algebra

Theorem (F., Schindler; ~2024)

Let V be a model of ZFC and let S be a finite set of mutually generic Cohen reals over V. Consider F the minimum field containing $\bigcup_{\substack{T \subseteq S \\ |T| = |S|-1}} \mathbb{R}^{V[T]}$.

Then the transcendence degree of $\mathbb{R}^{V[S]}$ over F is $\mathfrak{c}$.

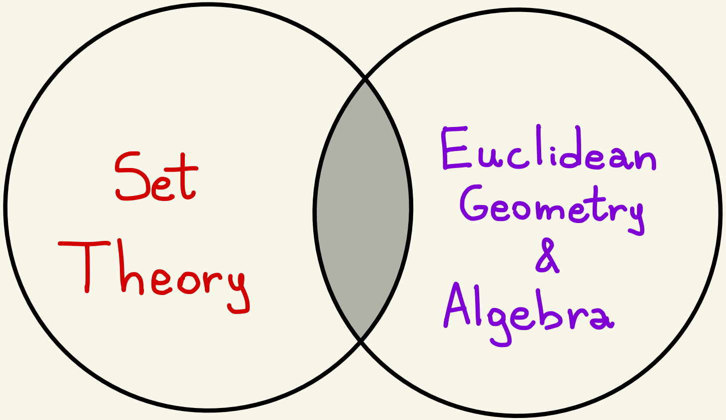


Summary

Recipes
$$\left\{\begin{array}{|l|}\hline ZF + DC + \neg WO(\mathbb{R}) + \psi(\mathcal{P}) \\ \hline ZF + DC + \neg \mathrm{Ul}(\omega) + \psi(\mathcal{P}) \\ \hline \end{array}\right.$$

Hamel basis
Mazurkiewicz set
PUC

Hamel basis



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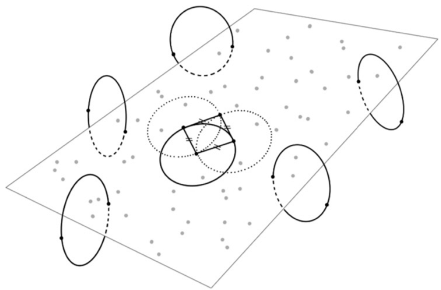

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