Higher models of determinacy

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Errata: Add the following sentence to the acknowledgements of the thesis.

- I thank Gabriel Goldberg for being a godfather of the thesis.

The related research area, determinacy and inner model theory, owes a great deal to W. Hugh Woodin, one of the guests in Young Academy Distinguished Lecture tomorrow. You might be surprised how many times I'll say "Woodin" in this talk (otherwise you are a set theorist).

Let X be any set (usually ω or $\mathbb{R} := \omega^\omega$) and let $A \subseteq X^\omega$. The game $G_\omega(A)$ on X of length ω with payoff set A is defined as follows: two players take turns elements of X.

$$\begin{array}{c||cccccc} \mathrm{I} & x_0 & & x_2 & & \cdots & \\ \hline \mathrm{II} & & x_1 & & x_3 & & \cdots \end{array}$$

I wins the game if and only if $\langle x_i \mid i < \omega \rangle \in A$.

A game is determined if one of the players has a winning strategy in the game.

Definition

AD_X *is the assertion that the games* $G_\omega(A)$ *are determined for all* $A \subseteq X^\omega$*. In particular,* $\mathrm{AD} := \mathrm{AD}_\omega$ *is called the* **Axiom of Determinacy***.*

- ZF + AD proves that all sets of reals have regularity property, such as Lebesgue measurability, the Baire property, and the perfect set property.
- ZFC proves $\neg$AD.
- Assuming ZFC plus the existence of enough large cardinals, AD holds in $L(\mathbb{R})$, the minimal inner model of ZF including all reals.

Main theme of my thesis

The interplay of large cardinal axioms and strong forms of AD is one of the central topics in set theory, especially in inner model theory.

My thesis "Higher model of determinacy" consists of three parts:

1. Derived models of self–iterable universes.
2. Determinacy in the Chang model.
3. $\mathsf{AD}_{\mathbb{R}} + \Theta$ is a strong cardinal.

All the main theorems concern how to get models of strong forms of AD using hod mice with large cardinals.

HOD: the class of Hereditarily Ordinal Definable sets

V is the class of all sets. Recall that $V = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$, where V_α is defined by transfinite recursion:

$$V_0 = \emptyset, V_{\alpha+1} = P(V_\alpha), \text{ and if } \lambda \text{ is limit, } V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha.$$

A set X is ordinal definable if X is defined by a formula with ordinal parameters, i.e.

$$x \in X \iff V_\alpha \models \phi[x, \beta_0, \cdots, \beta_{n-1}]$$

for some formula ϕ and some ordinals $\alpha, \beta_0, \ldots, \beta_{n-1}$.

A set X is hereditarily ordinal definable if the transitive closure of X

$$X \cup (\bigcup X) \cup (\bigcup \bigcup X) \cup \cdots$$

is ordinal definable. HOD **is the class of all hereditarily ordinal definable sets.**

Gödel showed that HOD is a model of ZFC.

Large cardinals

Recall that $j\colon M \to N$ is an elementary embedding if for any formula (in the language of set theory) $\phi(x_0, \ldots, x_{n-1})$ and for any $a_0, \ldots, a_{n-1} \in M$,

$$M \models \phi[a_0, \ldots, a_{n-1}] \iff N \models \phi[j(a_0), \ldots, j(a_{n-1})].$$

The critical point of j, denoted by $\mathrm{crit}(j)$, is the least ordinal α such that $j(\alpha) > \alpha$.

Large cardinals are often defined by the existence of an elementary embedding $j\colon V \to M$.

Definition (in ZFC)

- *A cardinal κ is a* **measurable cardinal** *if* $\exists j\colon V \to M\,(\mathrm{crit}(j) = \kappa)$.
- *A cardinal κ is a* **strong cardinal** *if* $\forall \beta\, \exists j\colon V \to M\,(\mathrm{crit}(j) = \kappa \wedge V_\beta \subseteq M)$.
- *A cardinal δ is a* **Woodin cardinal** *if*

$$\forall A \subseteq V_\delta\, \exists \kappa < \delta\, \forall \beta < \delta\, \exists j\colon V \to M\,(\mathrm{crit}(j) = \kappa \wedge j(A) \cap V_\beta = A \cap V_\beta).$$

A Woodin cardinal may not be measurable, but there are unboundedly many measurable cardinals below it.

Large cardinals in HOD

Under AD^+, Woodin cardinals can be naturally found in HOD.

Theorem (Woodin)

Assume $\mathsf{ZF} + \mathsf{AD}^+$. *Then*

$$\Theta := \sup\{\alpha \in \mathrm{Ord} \mid \exists f : \omega^\omega \to \alpha \, (f \text{ is surjective})\}$$

is either a Woodin cardinal or a limit of Woodin cardinal in HOD.

In ZFC context, there is a famous result known as Woodin's HOD dichotomy.

Theorem (Woodin)

If κ is an extendible cardinal, exactly one of the following holds:

1. *For all singular cardinals $\lambda > \kappa$, λ is singular in* HOD *and* $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$.
2. *All regular cardinals $\geq \kappa$ is measurable in* HOD.

Woodin's HOD conjecture states that the first item is provable from large cardinals.

Analysis of HOD in models of AD

The inner model theoretic analysis of HOD in models of AD was initiated by John R. Steel. It has been central theme in inner model theory over 30 years.

- Steel and Woodin showed that if AD holds in $L(\mathbb{R})$, then $\mathrm{HOD}^{L(\mathbb{R})}$ is a very nice inner model with a Woodin cardinal, satisfying GCH and other combinatorial properties.
- Sargsyan extended their results to much bigger models of AD by introducing and developing the theory of **hod mice**.
- Steel introduced another kind of hod mice and show the following.

Theorem (Steel)

Assume $\mathrm{ZF} + \mathrm{AD}_{\mathbb{R}} + \mathrm{HPC}$ *(Hod Pair Capturing). Then* $\mathrm{HOD} \cap V_\Theta$ *is a hod mouse.*

Sargsyan announced the result that HPC follows from AD^+ if there is no inner models of a Woodin limit of Woodin cardinals.

Theorem (Woodin)

Assume that $\mathrm{ZF} + \mathrm{AD}^+ + V = L(P(\mathbb{R}))$. *Then* HOD *satisfies Goldberg's Ultrapower Axiom.*

The Ultrapower Axiom is expected to hold in all canonical inner models with large cardinals.

Derived models

Woodin's derived model construction is a canonical way to get models of AD from large cardinals. If λ is a limit of Woodin cardinals, then the derived model at λ, denoted by DM_λ, always satisfies AD^+.

Also, a derived model can satisfy stronger determinacy axioms:

- If λ is also a limit of $< \lambda$-strong cardinals, then $\mathrm{DM}_\lambda \models \mathrm{AD}_\mathbb{R}$.
- If $\exists\kappa < \lambda$ (κ is λ-supercompact), then $\mathrm{DM}_\lambda \models$"$\mathrm{AD}_\mathbb{R} + \Theta$ is regular."

Theorem (G.–Sargsyan)

Assume that V is "self-iterable." If λ is a regular limit of Woodin cardinals, then DM_λ satisfies $\mathrm{AD}_\mathbb{R} + \Theta$ is regular.

Corollary

A derived model at a regular limit of Woodin cardinals of a hod mouse satisfies $\mathrm{AD}_\mathbb{R} + \Theta$ is regular.

This corollary was already proved by Sargsyan, but our proof can be used to show that "generalized" derived models satisfy $\mathrm{AD}_\mathbb{R} + \Theta$ is regular.

Chang model

The Chang model is the minimal inner model of ZF closed under countable sequences.

$$\mathsf{CM} := \bigcup_{\alpha \in \mathsf{Ord}} L(\alpha^\omega).$$

This model is not a derived model, but can satisfy determinacy:

Theorem (Woodin)

Assume that there are unboundedly many Woodin cardinals that are limit of Woodin cardinals. Then AD^+ holds in the Chang model.

(Mitchell) The Chang model cannot satisfy $\mathsf{AD}_{\mathbb{R}}$.

Theorem (G.–Sargsyan)

Let M be a countable excellent hod mouse with universally Baire iteration strategy such that

$$M \models \mathsf{ZFC} + \exists \text{Woodin cardinal that is limit of Woodin cardinals.}$$

Then AD^+ holds in the Chang model.

Our proof makes use of the Chang model over the derived model (CDM), which can be seen as a "generalized" derived model of a hod mouse.

Additional remarks on CDM

Roughly speaking, CDM_λ is defined by adding many, but not all, ω-sequences of ordinals to DM_λ. This model can be only defined in a (symmetric extension of) hod mouse.

Our theorem on derived models of self-iterable universes can be generalized to show that if λ is a regular limit of Woodin cardinals of a hod mouse, then CDM_λ satisfies $\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular.

This fact is used in the proof of the following theorem, which was obtained by studying the $\mathbb{P}_{\max}$ extension of CDM:

Theorem (Aksornthong, G., Holland & Sargsyan)

It is consistent relative to a Woodin limit of Woodin cardinals that ZFC *and for all* $\kappa \in \{\omega_1, \omega_2, \omega_3\}$,

the restriction of the club filter on $\kappa \cap \mathrm{Cof}(\omega)$ *to* HOD *is an ultrafilter in* HOD,

where $\mathrm{Cof}(\omega)$ *is the class of all ordinals of countable cofinality.*

This statement was motivated by Woodin's HOD conjecture—the above conclusion implies that κ is measurable in HOD.

Generalized Chang models

Woodin also introduced variants of the Chang model, e.g.

$$\mathsf{CM}^+ = \bigcup_{\alpha \in \mathsf{Ord}} L({}^{\omega}\alpha)[\mu_\alpha],$$

where μ_α is the club filter on $\wp_{\omega_1}(\alpha^\omega)$.

Theorem (Woodin)

Assume that there are unboundedly many Woodin cardinals that are limit of Woodin cardinals. Then CM^+ satisfies $\mathsf{AD}^+ + \omega_1$ is supercompact.

(Ikegami & Trang) Assume $\mathsf{ZF} + \omega_1$ is supercompact. Then AD^+ is equivalent to $\mathsf{AD}_{\mathbb{R}}$.

Theorem (Steel)

Let M be a countable excellent hod mouse with universally Baire iteration strategy such that $M \models \mathsf{ZFC} + \exists$measurable Woodin cardinal. Then CM^+ satisfies $\mathsf{AD}^+ + \omega_1$ is supercompact.

Steel defined CDM^+ as a variant of CDM, and use G.–Sargsyan's argument to prove the above theorem. Independently, G.–Müller–Sargsyan also introduced CDM^+ in a different method.

$\mathsf{AD}_{\mathbb{R}} + \Theta$ is strong

Atmai–Sargsyan analyzed HOD in a minimal model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable." In the thesis, I introduced the theory Θstrong := "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is a strong cardinal" and proved its consistency relative to a Woodin limit of Woodin cardinal.

Conjecture

Assume that Θstrong *is consistent. Then*

1. *There is a minimal model of* Θstrong.
2. HOD *in a minimal model of* Θstrong *is a hod mouse and* Θ *is strong in* HOD.

Here, we say that M is a minimal model of Θstrong if M is a transitive model of Θstrong containing all ordinals and reals, and no transitive $N \subsetneq M$ containing all ordinals and reals is a model of Θstrong.

The natural model of Θstrong has the form $L(P(\mathbb{R}))[\vec{E}]$, where $\vec{E}$ is a sequence of extenders. Determinacy models of such form are still not well-studied.

Other higher models of determinacy

Different kinds of models of AD derived from a hod mouse are used in the context of forcing over determinacy models.

Theorem

The following theories are consistent relative to a Woodin limit of Woodin cardinals.

1. *(Blue–Larson–Sargsyan)* $\mathsf{ZFC} + \forall i < n\,(\neg\square(\omega_{2+i}) + \neg\square_{\omega_{2+i}})$.
2. *(Schindler–Yasuda)* $\mathsf{ZFC} + \mathsf{MM}^{*,++}_{\mathfrak{c}}$.

Long games (games of length $> \omega$) can be a great resource to find new models of AD.

- Actually, Woodin proved determinacy in generalized Chang models by using long game determinacy.
- Determinacy of long games with definable payoff should be equivalent to the existence of canonical inner models with large cardinals. To prove such equivalence, we have to connect long game determinacy with determinacy in some model of AD.

More and more theorems on higher models of determinacy will follow in the near future.

Thank you for your attention!

- Takehiko Gappo & Grigor Sargsyan, *On derived models of self-iterable universes*, Proc. Amer. Math. Soc. 150 (2022), 1321-1329.
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