Proof mining and asymptotic regularity

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(joint work with Paulo Firmino)

Asymptotic regularity

(X, d) metric space, $C \subseteq X$, $T : C \to C$

- The notion of asymptotic regularity was introduced by Browder, Petryshyn (1967): T is asymptotically regular if

$$\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0 \quad \text{for all } x \in C$$

- Borwein, Reich, Shafrir (1992) adapted it to the Mann iteration.

Asymptotic regularity

For a general sequence (x_n), asymptotic regularity can be defined in two ways:

- (x_n) is asymptotically regular if

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$$

  rate of asymptotic regularity = rate of convergence of $\big(d(x_n, x_{n+1})\big)$ towards 0

- (x_n) is T-asymptotically regular if

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0$$

  rate of T-asymptotic regularity = rate of convergence of $\big(d(x_n, Tx_n)\big)$ towards 0

- $x_n = T^n x \Rightarrow x_{n+1} = Tx_n$

Asymptotic regularity

- In many cases, the first step towards proving weak or strong convergence of (x_n) is proving $(T$-$)$asymptotic regularity.
- Usually, one proves first that (x_n) is asymptotically regular and afterwards that (x_n) is T-asymptotically regular.
- $(T$-$)$asymptotic regularity is obtained in very general settings.
- Proof mining is very successful in computing rates of $(T$-$)$asymptotic regularity.

Asymptotic regularity

The notion of T-asymptotic regularity can be extended to countable families of mappings.

(X, d) metric space, $C \subseteq X$, $T_n : C \to C$ for all $n \in \mathbb{N}$, (x_n)

- (x_n) is (T_n)-asymptotically regular if

$$\lim_{n\to\infty} d(x_n, T_n x_n) = 0.$$

rate of (T_n)-asymptotic regularity = rate of convergence of $\big(d(x_n, T_n x_n)\big)$ towards 0

Quantitative notions

(a_n) sequence in a metric space (X, d), $a \in X$ and $\varphi : \mathbb{N} \to \mathbb{N}$

- Assume that $\lim_{n\to\infty} a_n = a$, i.e.

$$\forall k \in \mathbb{N}\, \exists N \in \mathbb{N}\, \forall n \geq N \left(d(a_n, a) \leq \frac{1}{k+1} \right).$$

  Then φ is a rate of convergence for (a_n) (towards a) if

$$\forall k \in \mathbb{N}\, \forall n \geq \varphi(k) \left(d(a_n, a) \leq \frac{1}{k+1} \right).$$

- Assume that (a_n) is Cauchy, i.e.

$$\forall k \in \mathbb{N}\, \exists N \in \mathbb{N}\, \forall m, n \geq N \left(d(a_m, a_n) \leq \frac{1}{k+1} \right).$$

  Then φ is a Cauchy modulus for (a_n) if

$$\forall k \in \mathbb{N}\, \forall m, n \geq \varphi(k) \left(d(a_m, a_n) \leq \frac{1}{k+1} \right).$$

Quantitative notions

(b_n) sequence of nonnegative real numbers

- Assume that $\sum_{n=0}^{\infty} b_n$ diverges, i.e.

$$\forall n \in \mathbb{N}\, \exists K \in \mathbb{N} \left(\sum_{i=1}^{K} b_i \geq n \right).$$

  Then a rate of divergence of the series is a function $\theta : \mathbb{N} \to \mathbb{N}$ satisfying $\forall n \in \mathbb{N} \sum_{i=0}^{\theta(n)} b_i \geq n$.

- A Cauchy modulus of a convergent series $\sum_{n=0}^{\infty} b_n$ is a Cauchy modulus of the sequence $\left(\sum_{i=0}^{n} b_i \right)$ of partial sums.

W-spaces

A W-space is a structure (X, d, W), where (X, d) is a metric space and $W : X \times X \times [0,1] \to X$ is a function.

- W is called a convexity mapping; already considered by Takahashi in the 1970s.
- We use the notation $(1-\lambda)x + \lambda y$ for $W(x, y, \lambda)$.

- For all $x, y \in X$,

$$[x, y] = \{(1-\lambda)x + \lambda y \mid \lambda \in [0,1]\}$$

- $C \subseteq X$ is convex if $[x, y] \subseteq C$ for all $x, y \in C$.

W-hyperbolic spaces

A W-hyperbolic space is a W-space (X, d, W) satisfying the following for all $x, y, w, z \in X$ and all $\lambda,\ \tilde{\lambda} \in [0,1]$:

$$\begin{aligned}
&\text{(W1)} \quad d(z,(1-\lambda)x+\lambda y) \le (1-\lambda)d(z,x)+\lambda d(z,y),\\
&\text{(W2)} \quad d((1-\lambda)x+\lambda y,(1-\tilde{\lambda})x+\tilde{\lambda}y) = |\lambda-\tilde{\lambda}|d(x,y),\\
&\text{(W3)} \quad (1-\lambda)x+\lambda y = \lambda y+(1-\lambda)x,\\
&\text{(W4)} \quad d((1-\lambda)x+\lambda z,(1-\lambda)y+\lambda w) \le (1-\lambda)d(x,y)+\lambda d(z,w).
\end{aligned}$$

- introduced by Kohlenbach (2005) under the name of "hyperbolic spaces"
- also called Kohlenbach hyperbolic spaces

- (X, d, W) is a geodesic space: for all $x, y \in X$, $[x, y]$ is a geodesic segment joining x and y.

W-hyperbolic spaces

Examples:

- normed spaces: $W(x,y,\lambda) = (1-\lambda)x + \lambda y$
- Busemann spaces = uniquely geodesic W-hyperbolic spaces
- CAT(0) spaces = W-hyperbolic spaces satisfying, for all $x, y, z \in X$,

$$d^2\left(z, \frac{1}{2}x + \frac{1}{2}y\right) \leq \frac{1}{2}d^2(z,x) + \frac{1}{2}d^2(z,y) - \frac{1}{4}d^2(x,y)$$

Generalized VAM iteration

Generalized VAM iteration

X is a W-hyperbolic space, C is a nonempty convex subset of X, $f : C \to C$ is an α-contraction for $\alpha \in [0,1)$, $(\alpha_n) \subseteq [0,1]$, $(T_n : C \to C)_{n\in\mathbb{N}}$ is a countable family of nonexpansive mappings such that $\bigcap_{n\in\mathbb{N}} Fix(T_n) \neq \emptyset$.

The generalized Viscosity Approximation Method (genVAM) is defined by Firmino, L. (2024) as follows:

$$\text{genVAM} \quad x_0 = x \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T_n x_n.$$

- X normed space, $C = X$, A m-accretive operator on X, $(\lambda_n) \subseteq (0,\infty)$, $T_n = J^A_{\lambda_n}$ (resolvent of order λ_n of A) $\Longrightarrow$ genVAM becomes Viscosity Approximation Method (VAM), studied by Xu et al. (J. Nonlinear Var. Anal. 2022).
- Proof mining applied by Firmino, L. (Z. Anal. Anwend. 2024) to obtain quantitative asymptotic regularity results for VAM with error terms.

Generalized VAM iteration

X is a W-hyperbolic space, C is a nonempty convex subset of X, $f : C \to C$ is an α-contraction for $\alpha \in [0,1)$, $(\alpha_n) \subseteq [0,1]$, $(T_n : C \to C)_{n\in\mathbb{N}}$ is a countable family of nonexpansive mappings such that $F := \bigcap_{n\in\mathbb{N}} Fix(T_n) \neq \emptyset$.

The generalized Viscosity Approximation Method (genVAM) is defined as follows:

$$\text{genVAM} \quad x_0 = x \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T_n x_n.$$

- f is constant (hence $\alpha = 0$) $\implies$ genVAM becomes the Halpern-type abstract proximal point algorithm studied by Sipoș (Comput. Optim. Appl. 2022).

Generalized VAM iteration

For every sequence (λ_n) of positive reals and $m, n \in \mathbb{N}$, we say that the family (T_n) satisfies $Res((\lambda_n), n, m)$ if

$$d(T_n y, T_m y) \leq \frac{|\lambda_n - \lambda_m|}{\lambda_n} d(y, T_n y) \text{ for all } y \in C.$$

- $Res((\lambda_n), n, m)$ was introduced in L.,Nicolae, Sipoș (J. Global Optim. 2018) to obtain abstract versions of the proximal point algorithm.
- The family $(J^A_{\lambda_n})$ of resolvents of m-accretive operators satisfies $Res((\lambda_n), n, m)$ for all $m, n \in \mathbb{N}$.

Quantitative hypotheses on the parameter sequences

$(H1\alpha_n)$ $\sum_{n=0}^{\infty} \alpha_n = \infty$ with rate of divergence σ_1;

$(H2\alpha_n)$ $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ with Cauchy modulus σ_2;

$(H3\alpha_n)$ $\lim_{n\to\infty} \alpha_n = 0$ with rate of convergence σ_3;

$(H1\lambda_n)$ $\sum_{n=0}^{\infty} \left|1 - \frac{\lambda_{n+1}}{\lambda_n}\right| < \infty$ with Cauchy modulus γ_1;

$(H2\lambda_n)$ $\Lambda \in \mathbb{N}^*$ and $N_\Lambda \in \mathbb{N}$ are such that $\lambda_n \geq \frac{1}{\Lambda}$ for all $n \geq N_\Lambda$;

genVAM - rates of asymptotic regularity

$z \in \bigcap_{n\in\mathbb{N}} Fix(T_n)$, $(H1\alpha_n)$, $(H2\alpha_n)$, $(H1\lambda_n)$ hold,

$$K_z \in \mathbb{N}^* \text{ such that } K_z \geq \max\left\{d(x,z), \tfrac{d(f(z),z)}{1-\alpha}\right\},$$

$$\chi(k) = \max\{\sigma_2(4K_z(k+1)-1), \gamma_1(4K_z(k+1)-1)\}.$$

Theorem (Firmino, L. (2024))

Assume that (T_n) satisfies $Res((\lambda_n), n, n+1)$ for every $n \in \mathbb{N}$. Then (x_n) is asymptotically regular with rate

$$\Phi(k) = \sigma_1\left(\left\lceil \tfrac{\chi(2k+1)+1+\lceil \ln(4K_z(k+1))\rceil}{1-\alpha}\right\rceil + 1\right).$$

- Φ has a very weak dependency on X, C, f, (T_n):
  only via K_z
- If C bounded with diameter d_C, then take $K_z = \left\lceil \frac{d_C}{1-\alpha} \right\rceil$.

Proposition (Firmino, L. (2024))

Φ is a rate of asymptotic regularity of (x_n),
$(H3\alpha_n)$ holds.
Then (x_n) is (T_n)-asymptotically regular with rate

$$\Psi(k) = \max\{\sigma_3(4K_z(k+1)-1), \Phi(2k+1)\}.$$

Proposition (Firmino, L. (2024))

$m \in \mathbb{N}$, $\Lambda_m \in \mathbb{N}^*$ is such that $\Lambda_m \geq \lambda_m$,
Ψ is a rate of (T_n)-asymptotic regularity of (x_n),
$(H2\lambda_n)$ holds,
(T_n) satisfies $Res((\lambda_n), n, m)$ for every $n \in \mathbb{N}$.
Then (x_n) is T_m-asymptotically regular, with rate

$$\Theta(k) = \max\{N_\Lambda, \Psi(\Lambda_m\Lambda(k+1)-1), \Psi(2k+1)\},$$

genVAM - $((T_n)$-, T_m-)asymptotic regularity

Theorem

Assume that (T_n) satisfies $Res((\lambda_n), n, m)$ for every $m, n \in \mathbb{N}$ and that

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \qquad \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty, \quad \lim_{n\to\infty} \alpha_n = 0,$$

$$\sum_{n=0}^{\infty} \left|1 - \frac{\lambda_{n+1}}{\lambda_n}\right| < \infty, \qquad \liminf_{n\in\mathbb{N}} \lambda_n > 0.$$

Then

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, T_n x_n) = 0$$

and, for every $m \in \mathbb{N}$,

$$\lim_{n\to\infty} d(x_n, T_m x_n) = 0.$$

genVAM - linear rates

Proposition

(T_n) satisfies $Res((\lambda_n), n, m)$ for every $m, n \in \mathbb{N}$,

$$\alpha_n = \frac{2}{(1-\alpha)(n+J)}, \quad \lambda_n = \frac{n+J}{n+J-1}$$

where $J = 2\left\lceil \frac{1}{1-\alpha} \right\rceil$. Let $M = \frac{3K_z J^2}{2}$ and define

$$\Phi_0(k) = M(k+1)-J, \quad \Psi_0(k) = 3M(k+1)-J, \quad \Theta_0(k) = 6M(k+1)-J.$$

Then Φ_0 is a linear rate of asymptotic regularity of (x_n), Ψ_0 is a linear rate of (T_n)-asymptotic regularity of (x_n), and Θ_0 is a linear rate of T_m-asymptotic regularity of (x_n) for every $m \in \mathbb{N}$.

- Application of a lemma on sequences of real numbers due to Sabach, Shtern (2017).

Immediate consequences

- Rates for VAM:

$$\text{VAM} \qquad x_0 = x \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J^A_{\lambda_n} x_n$$

identical with the rates obtained by Firmino, L. (Z. Anal. Anwend. 2024)

- Rates for the Halpern-type abstract proximal point algorithm:

$$\text{VAM} \qquad x_0 = x \in X, \quad x_{n+1} = \alpha_n u + (1-\alpha_n) T_n x_n$$