O-minimal definitions of the complex Γ and ζ functions

Adele Padgett - University of Vienna - Colloquium Logicum 2024
(Joint with Patrick Speissegger)

Plan ① Background on Γ and ζ

② Explanation, context, discussion of

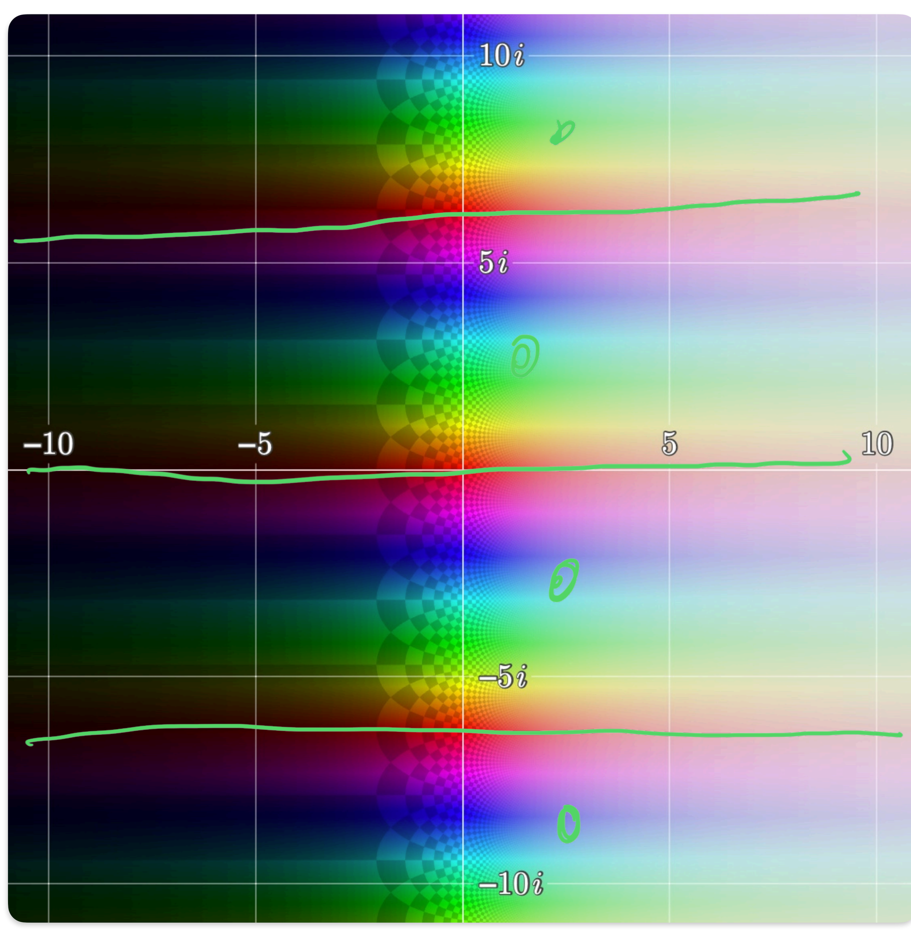
Thm (P., Speissegger): Γ restricted to certain unbounded domains in $\mathbb{C}$ is o-minimal.

③ **Thm**: ζ restricted to certain unbounded domains in $\mathbb{C}$ is o-minimal.

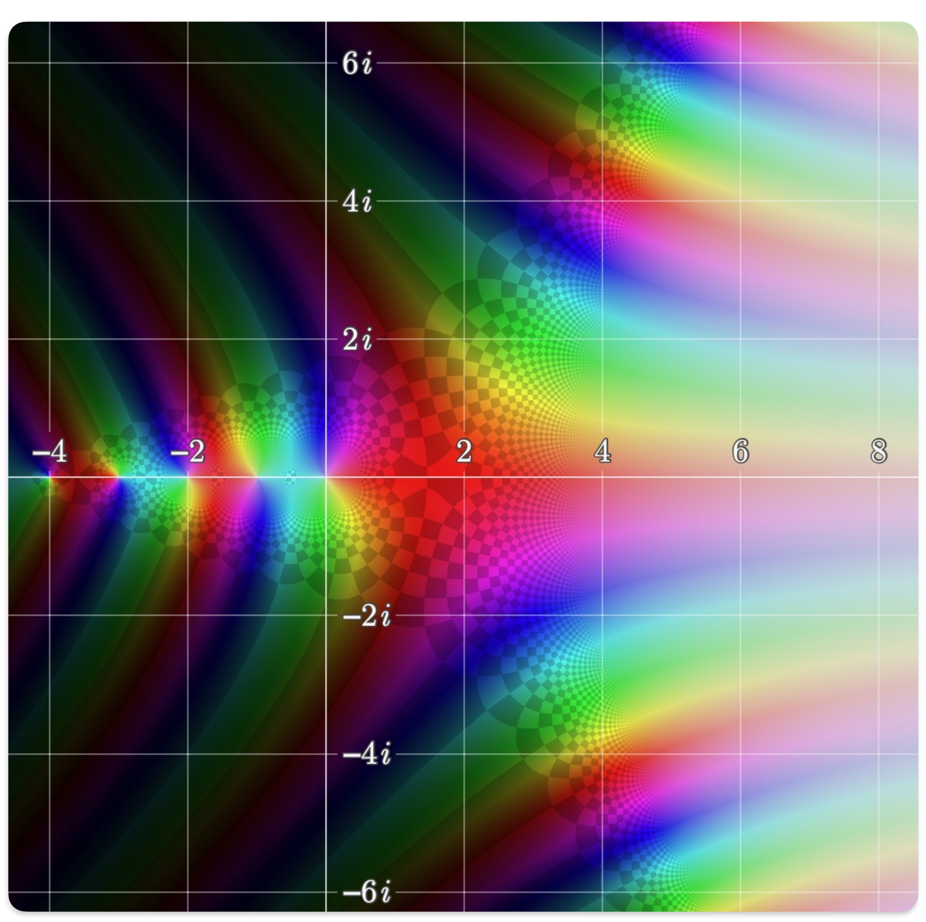
Motivations · Use o-minimality to prove functional transcendence results
· O-minimal definability of holomorphic extensions

① Background

- exp is an entire function
- $\exp(z) = \exp(z + 2\pi i n)$, $n \in \mathbb{Z}$



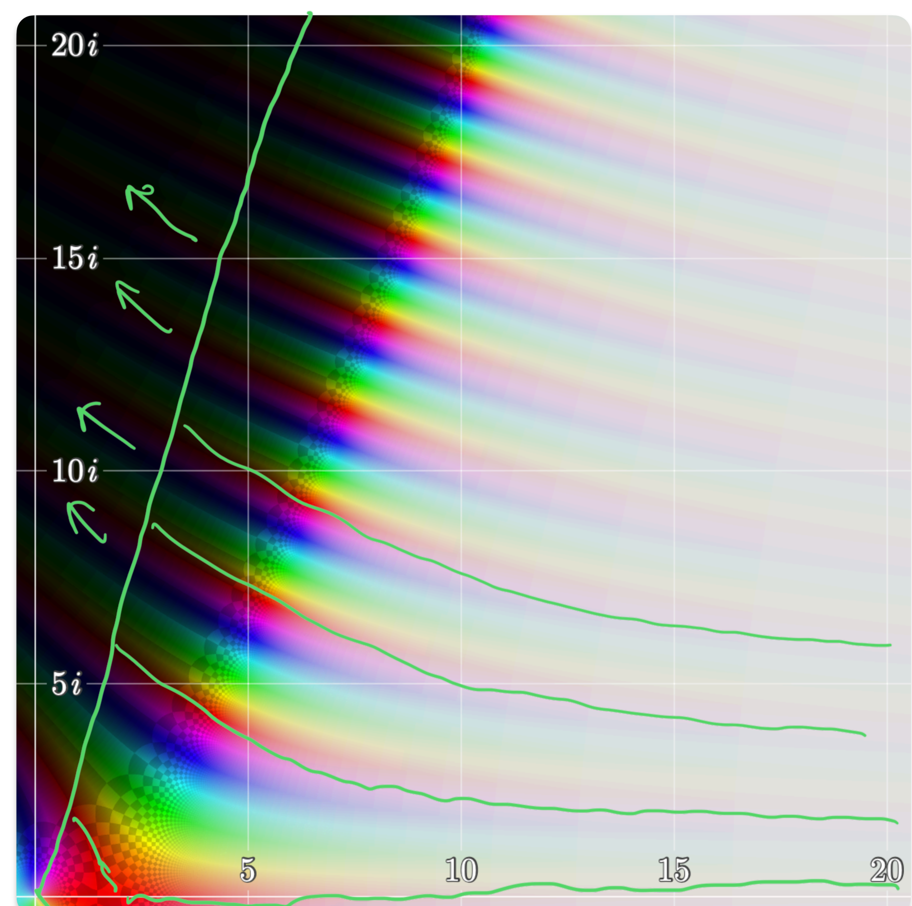
- Thm (Wilkie): $\mathbb{R}_{(\exp|_{\mathbb{R}})}$ is o-minimal
- Thm (van den Dries, Miller): $\mathbb{R}_{an, \exp|_{\mathbb{R}}}$ is o-minimal and defines exp restricted to horizontal strips with bounded imaginary part.
- Many applications (e.g. to functional transcendence, Diophantine geometry) use complex definability.



https://samuelj.li/complex-function-plotter/#gamma(z)

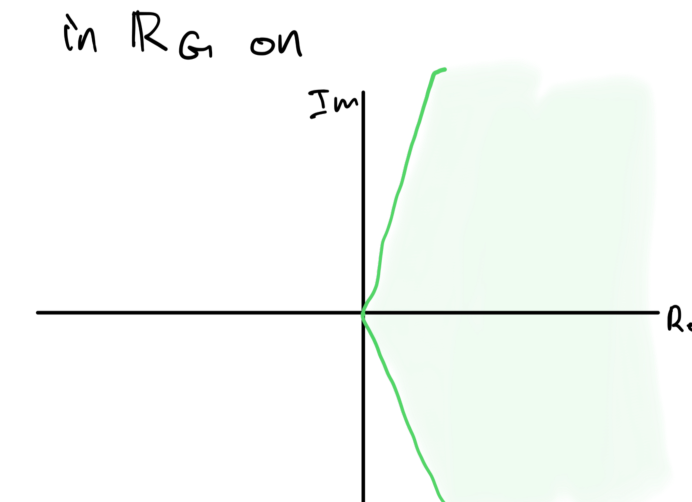
- Γ is a meromorphic function, holomorphic on $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$
- $\Gamma(1) = 1$, $\Gamma(z+1) = z\Gamma(z)$
  $\Gamma(n+1) = n!$
- $\Gamma(z) = \sqrt{2\pi}\, e^{(z-\frac{1}{2})\log z - z + \varphi(z)}$ on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$
  Stirling's function
- Thm (van den Dries, Speissegger): $\Gamma|_{(0,\infty)}$ is definable in the o-minimal structure $\mathbb{R}_{G, \exp}$.
  $\varphi|_{(0,\infty)}$

② Defining complex Γ o-minimally



Thm (P., Speissegger): If $\varphi(x) \in G$, then $\mathrm{Re}(\varphi)(r,\theta)$, $\mathrm{Im}(\varphi)(r,\varphi) \in G$.

ex. Stirling function Φ is definable in $\mathbb{R}_G$ on



· $\Gamma(z) = \sqrt{2\pi}\, e^{(z-\frac{1}{2})\log z - z + \varphi(z)}$

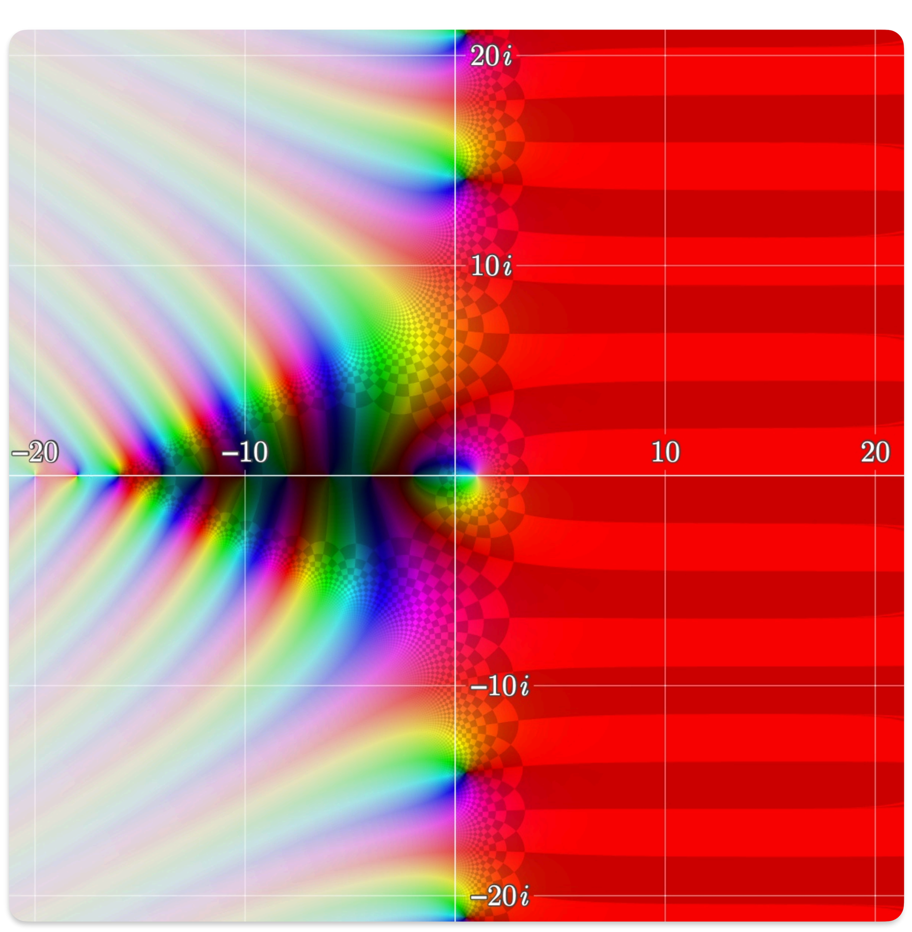
Cor (P., Speissegger): The real and imaginary parts of Γ restricted to any strip are definable in $\mathbb{R}_{G, \exp|_{\mathbb{R}}}$.

Γ is not definable on any unbounded set in the left half plane. in $\mathbb{R}_{G,\exp}$

Motivation

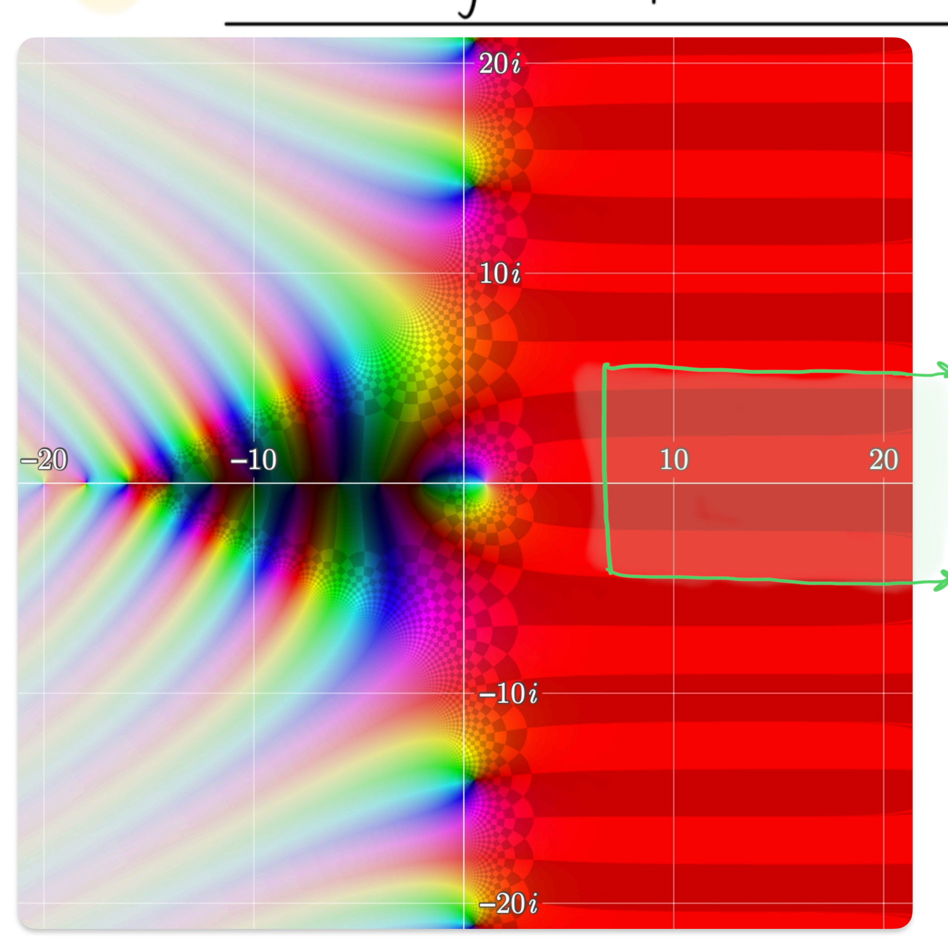
- Complex definability of Γ on unbounded domains may be useful in applications to functional transcendence.
- Kaiser + Speissegger studied when functions definable in o-minimal structures have definable holomorphic extensions.

- ζ is a meromorphic function holomorphic on $\mathbb{C} \setminus \{1\}$
- For $\mathrm{Re}(z) > 1$, $\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$,
  a convergent series



- $\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\zeta(1-z)$
- Thm (van den Dries, Speissegger): $\zeta|_{(1,\infty)}$ is definable in the o-minimal structure $\mathbb{R}_{an^*, \exp}$
  $\zeta(-\log x)$

③ Defining complex ζ o-minimally



can be a tuple

Thm (P., Speissegger): If $\varphi(x) \in an^*$, then $\mathrm{Re}(\varphi)(r,\theta)$, $\mathrm{Im}(\varphi)(r,\theta) \in an^*$.

ex. $\zeta(-\log z) = \sum_{n=1}^{\infty} \frac{1}{n^{-\log z}} = \sum_{n=1}^{\infty} z^{\log n}$

is definable in $\mathbb{R}_{an^*}$ on $[0,r] \times [-s,s]$ if $r < e^{-(s+1)}$

Cor: For every $s > 0$, ζ is definable in $\mathbb{R}_{an^*, \exp}$ on $\{z : \mathrm{Re}(z) > s+1, |\mathrm{Im}(z)| < s\}$.