Sunny nonexpansive retractions in nonlinear spaces

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Colloquium Logicum 2024
Austrian Academy of Sciences

October 7–9, 2024

Outline

1. Generalization of proofs
2. Smooth W-hyperbolic spaces
3. Results

Proof mining

Inspired by Kreisel's program of unwinding of proofs (1950s),

> *"(...) what more we know about a formally derived theorem* F *than if we merely know that* F *is true?"*

Application of proof interpretations to study *a priori* noneffective mathematical proofs as a way to obtain:

- effective bounds, algorithms;
- uniformities in the parameters;
- weakening of premisses, generalization of proofs.

This talk is focused on the *"generalization of proofs"*.

Recent examples of generalizations

- "Lion-Man" game – weakening of compactness assumption;[1]
- Suzuki's theorem reducing the convergence of a generalized iterative schema to that of its original version;[2]
- Halpern-type abstract proximal algorithm in CAT(0);[3]
- Strong convergence of a general new iterative schema.[4]

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[4] B.Dinis and P.Pinto. Strong convergence for the alternating Halpern-Mann iteration in CAT(0) spaces. SIAM J. Optim., 33(2), 785–815, 2023.

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Successful in generalizations from the linear to the nonlinear setting.

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. . . in Hilbert spaces

Browder (1967)

Let X be a Hilbert space, $C \subseteq X$ be a nonempty closed convex subset of X. Let $T : C \to C$ be a nonexpansive map on C, and $u \in C$. For each $t \in (0,1)$, consider $z_t \in C$ characterized by

$$z_t = (1-t)T(z_t) + tu. \qquad \text{(B)}$$

If C is bounded and $t \to 0$, then $(z_t)_t$ converges strongly towards $P(u)$ where $P : C \to \mathrm{Fix}(T)$ is the metric projection onto $\mathrm{Fix}(T)$.

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For each $u \in C$, we have that $P(u)$ is the unique fixed point s.t.

$$\forall y \in \mathrm{Fix}(T)\,(\|u - P(u)\| \leqslant \|u - y\|)$$

which is equivalent to having for all $y \in \mathrm{Fix}(T)$

$$\langle u - P(u), y - P(u)\rangle = \langle y - P(u), u - P(u)\rangle \leqslant 0$$

. . . beyond Hilbert spaces

Let X be a (real) normed space. The duality map, $J : X \to 2^{X^*}$ is defined for all $x \in X$ by

$$J(x) := \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where X^* is the dual space of X and $\langle y, f\rangle$ denotes the functional application $f(y)$. J is homogeneous, i.e. $J(\alpha x) = \alpha J(x)$, for $\alpha \in \mathbb{R}$.

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where X^* is the dual space of X and $\langle y, f \rangle$ denotes the functional application $f(y)$. J is homogeneous, i.e. $J(\alpha x) = \alpha J(x)$, for $\alpha \in \mathbb{R}$. The space X is *smooth* if for any x, y with $\|x\| = \|y\| = 1$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \qquad (\star)$$

exists. We know that X is smooth iff J is single-valued. Moreover, X is *uniformly smooth* if the limit $(\star)$ is attained uniformly in x, y, in which case the duality map is also norm-to-norm uniformly continuous on bounded subsets.

. . . beyond Hilbert spaces

A normed space X is uniformly convex if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in \overline{B}_1(0) \left(\|x - y\| \geqslant \varepsilon \to \left\| \frac{x+y}{2} \right\| \leqslant 1 - \delta \right).$$

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Let X be unif. smooth and unif. convex, and C, E subsets of X with $E \neq \varnothing$ convex.

Metric projection $P : C \to E$

$$\forall y \in E\, (\langle y - P(u), J(u - P(u)) \rangle \leqslant 0)$$

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Sunny nonexpansive retraction $Q : C \to E$

$$\forall y \in E\, (\langle u - Q(u), J(y - Q(u)) \rangle \leqslant 0)$$

A celebrated result due to Reich extends Browder's theorem, proving in particular:

Reich (1980)

Let X be a unif. smooth and unif. convex Banach space, $C \subseteq X$ be a nonempty closed convex subset of X. Let $T : C \to C$ be a nonexpansive map on C and $u \in C$. For each $t \in (0,1)$, consider $z_t \in C$ satisfying (B). If C is bounded and $t \to 0$, then $(z_t)_t$ converges strongly towards $Q(u)$ where Q is the unique sunny nonexpansive retraction $Q : C \to \mathrm{Fix}(T)$.

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The proof-theoretical analysis of this result was obtained by Kohlenbach and Sipoș in 2021, probably the most complex proof mining analysis to date.[5,6]

[5] U.Kohlenbach and A.Sipoș. The finitary content of sunny nonexpansive retractions. Communications in Contemporary Mathematics, 23(1),63pp, 2021.

[6] A.Sipoș. On quantitative metastability for accretive operators. Zeitschrift für Analysis und ihre Anwendungen, 43(3-4): 417–433, 2024.

A triple (X, d, W) is a hyperbolic space (Kohlenbach) if (X, d) is a metric space and $W : X \times X \times [0, 1] \to X$ satisfies

W1 $d(W(x, y, \lambda), z) \leqslant (1 - \lambda)d(x, z) + \lambda d(y, z)$
W2 $d(W(x, y, \lambda), W(x, y, \lambda')) = |\lambda - \lambda'|d(x, y)$
W3 $W(x, y, \lambda) = W(y, x, 1 - \lambda)$
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We write $(1 - \lambda)x \oplus \lambda y$ for $W(x, y, \lambda)$.

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We write $(1 - \lambda)x \oplus \lambda y$ for $W(x, y, \lambda)$. A hyperbolic space is unif. convex if there is a function $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ s.t.

$$\left.\begin{aligned} d(x, a) &\leqslant r \\ d(y, a) &\leqslant r \\ d(x, y) &\geqslant \varepsilon \cdot r \end{aligned}\right\} \to d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leqslant (1 - \eta(r, \varepsilon))\, r$$

X is a *UCW* hyperbolic space (Leuștean) if η is nonincreasing in r.

In any metric space, the quasi-linearization function

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle := \frac{1}{2}\left(d^2(x,v) + d^2(y,u) - d^2(x,u) - d^2(y,v)\right)$$

is the unique function satisfying:

(1) $\langle \overrightarrow{xy}, \overrightarrow{xy} \rangle = d^2(x,y)$,
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A hyperbolic space is a CAT(0) space if

$$\text{(CS)} \quad \langle \overrightarrow{xy}, \overrightarrow{uv} \rangle \leqslant d(x,y) \cdot d(u,v)$$

[7] Any CAT(0) space is a UCW hyperbolic space w/ $\eta(r,\varepsilon) := \varepsilon^2/8$.

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Think of $\langle \overrightarrow{\cdot\,\cdot}, \overrightarrow{\cdot\,\cdot} \rangle$ as a nonlinear counterpart to an inner-product.

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We say that X is a smooth hyperbolic space if there is a function $\pi : X^2 \times X^2 \to \mathbb{R}$ satisfying

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Think of $\pi(\overrightarrow{\cdot}, \overrightarrow{\cdot})$ as a nonlinear counterpart to the duality map.

Uniformly smooth

The space X is a uniformly smooth hyperbolic space if additionally

P6
$$\forall \varepsilon > 0 \; \forall r > 0 \; \exists \delta > 0 \; \forall a \in X \; \forall u, v \in \overline{B}_r(a)$$
$$d(u,v) \leqslant \delta \to \forall x, y \in X \left(|\pi(\overrightarrow{xy}, \overrightarrow{ua}) - \pi(\overrightarrow{xy}, \overrightarrow{va})| \leqslant \varepsilon \cdot d(x,y) \right).$$

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- Extending the formal system $\mathcal{A}^\omega[X, d, W]$ with a new constant $\pi : 1(X)(X)(X)(X)$ satisfying (the universal) P1–P5, allows for a bound extraction theorem for results in smooth hyperbolic spaces.

- If we additionally include a modulus of uniform continuity for π, ω_X, providing a witnesses for δ in P6, we can also analyse results in unif. smooth hyperbolic spaces.

More than CAT(0) spaces

Proposition

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Proposition

Any (uniformly) smooth normed space is a (uniformly) smooth hyperbolic space, with $\pi(\overrightarrow{xy}, \overrightarrow{uv}) := \langle x - y, J(u - v)\rangle$.

- Therefore, the class of (unif.) smooth hyperbolic spaces properly extends the class of CAT(0) spaces, and we regard it as a nonlinear counterpart to (unif.) smooth normed spaces.

π-sunny nonexpansive retractions

Definition

Let X be a smooth hyperbolic space and $E \subseteq C$ subsets of X. A retraction $Q : C \to E$ is a (π-)sunny nonexpansive retraction if

$$\forall x \in C \ \forall y \in E \left(\pi\left(\overrightarrow{xQ(x)}, \overrightarrow{yQ(x)}\right) \leqslant 0 \right).$$

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Lemma

(1) *Any π-sunny nonexpansive retraction is a π-firmly n.e. map, i.e.*

$$\forall x, y \in C \left(d^2(Q(x), Q(y)) \leqslant \pi\left(\overrightarrow{xy}, \overrightarrow{Q(x)Q(y)}\right)\right),$$

and so, in particular, it is a nonexpansive map.

(2) *There exists at most one sunny nonexpansive retraction from C onto E.*

Nonlinear generalization of Reich's theorem

Relying on the proof-theoretically simpler proof due to Kohlenbach and Sipoș in the linear case, we obtained

Theorem (P. 2023)

Let X be a complete uniformly smooth UCW hyperbolic space, C a closed nonempty bounded convex subset, and $u \in C$. Consider $T : C \to C$ a nonexpansive map on C. For any $t \in (0, 1]$, let z_t denote the unique point in C satisfying $z_t = (1 - t)T(z_t) \oplus tu$. Then, for all $(t_n) \subseteq (0, 1]$ such that $\lim t_n = 0$, we have that (z_{t_n}) converges to a fixed point of T.

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If we set $Q(u) := \lim z_t$ (is well-defined and a retraction), then

Proposition

The map Q is the unique π-sunny nonexpansive retraction from C onto $\mathrm{Fix}(T)$.

Final Remarks

- Introduced the notion of smooth hyperbolic space: more general than CAT(0) spaces as well as smooth Banach spaces. The function π is a nonlinear version of the duality map of smooth Banach spaces. We have a formal system suitable for bound extraction in these spaces.

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- Proved the convergence of the Halpern's schema for a family of n.e. maps which have properties akin to families of resolvent maps when in a linear setting, extending/unifying previous results.

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- Proved the convergence of the Halpern's schema for a family of n.e. maps which have properties akin to families of resolvent maps when in a linear setting, extending/unifying previous results.
- Generalized a result by Chang: the necessary conditions (i),(ii) are sufficient for convergence, when one has asymptotic regularity.
- Further results extending Wittmann, Bauschke, and even with viscosity terms in the general sense of Meir-Keeler.

Some open questions

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- Are firmly nonexpansive maps (metrically characterized) always π-firmly nonexpansive?
- Are π-sunny nonexpansive retractions actually 'sunny'?
- ...

- P. Pinto. Nonexpansive maps in nonlinear smooth spaces. Transactions of the American Mathematical Society, 377(9): 6379–6426, 2024.
- P. Pinto, The finitary content of sunny nonexpansive retractions in nonlinear spaces. Manuscript in preparation, 2024.

Thank you for your attention!