(Conceptual) Explanations in Logic

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7 October 2024, Colloquium Logicum, Vienna, Austria

Aim of the talk

Explanation, the main object of this talk, has been one of the most intensely discussed topics of philosophy of science in the 20th century.

It is therefore useful to start with an orientation.

Several different types of explanations:

- explaining the meaning of a symbol,
- explaining a new concept to a child,
- explaining how to construct an Ikea furniture.

We only focus on deductive explanations why a phenomenon occurred/a proposition is true.

Causal Explanations

Causal Explanations $\rightarrow$ Causal Relation

Causal Explanations $\rightarrow$

Cause(s)[1]

$\vdots$

Phenomenon/Effect

[1] Both causes and effect are verified events or facts.

Causal Explanations $\rightarrow$

A cigarette lit in the forest

$\vdots$

A fire in the forest

Causal Explanations $\rightarrow$

Burning of fossil fuels

$\vdots$

Climate change

Conceptual Explanations

Conceptual Explanations $\rightarrow$ Grounding Relation

Conceptual Explanations $\rightarrow$

Ground(s) or reason(s)[1]

$\vdots$

Conclusion

[1] Both grounds and conclusion are true items.

Conceptual Explanations $\rightarrow$

That animal being a female and that animal being a fox

$$\vdots$$

That animal being a vixen

Conceptual Explanations $\rightarrow$

Jane being mathematically talented, Jane being a hard worker, Jane having logic interests....

$\vdots$

Jane being an ideal candidate for a post-doc in logic at Vienna University

Conceptual/Mathematical Explanations

Conceptual/Mathematical Explanations $\rightarrow$

For any two points x and y, there always exists a third z such that $l(xy) = l(yz) = l(xz)$.

$$\vdots$$

For any two circles X and Y, one with center in x and radius xy, the other with center in y and radius xy, there exists a point z where they intersect and which is such that $l(xy) = l(yz) = l(xz)$.

Bernard Bolzano, *Theory of Science*, 1837.

Although causal explanations are central in scientific inquiry and philosophy, logic has been argued to have a problematic relationship with causality.[1]

[1] E.g., see Scriven, M., The logic of cause, *Theory and Decision*, 2: 49-66, 1971. Note that quite recently there has been an opposite trend in the works of, e.g., Ibeling and Icard 2020, Moss, Ibeling, Icard, 2022.

As a result, as far as we know, there is no serious logical investigation of the concept of (causal) explanation.

Conceptual explanations naturally invite logical analysis.

This is precisely the aim of this talk: to elaborate *a logical theory of conceptual explanations*.

On the one hand, this will allow us to introduce the notion of explanation in logic, which is so far being a great absentee of the logical literature.

On the other hand, this will enlighten our understanding of conceptual explanations.

As a methodology, we will rely on a dialogue between philosophy and logic.

Formal framework

Causal Explanation $\longrightarrow$ Causality

Causal Explanation	$\longrightarrow$	Causality
Conceptual explanation	$\longrightarrow$	Grounding

Causal Explanation $\longrightarrow$ Causality

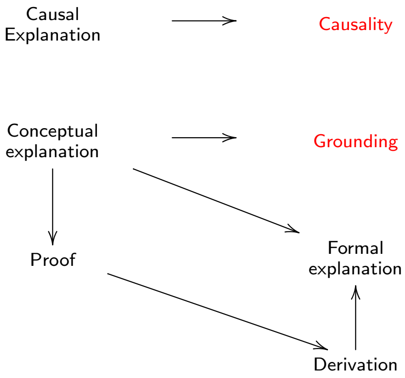
Conceptual explanation $\longrightarrow$ Grounding

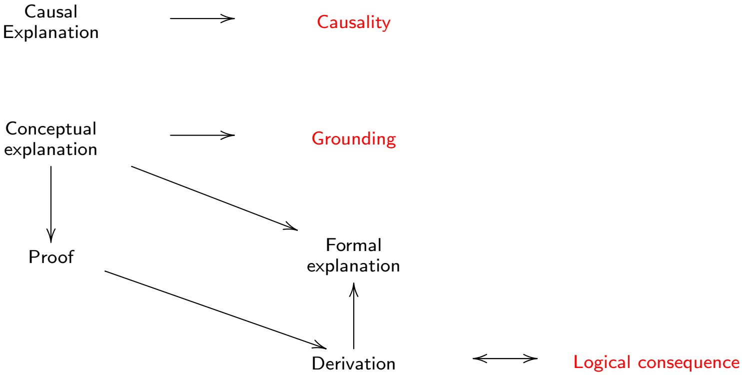
$$\begin{array}{ccc} \text{Causal Explanation} & \longrightarrow & \text{Causality} \\ \\ \text{Conceptual explanation} & \longrightarrow & \text{Grounding} \\ \downarrow & & \\ \text{Deductive argument} & & \end{array}$$

$$
\begin{tikzcd}
\text{Causal Explanation} \arrow[r] & \text{Causality} \\
\text{Conceptual explanation} \arrow[r] \arrow[d] & \text{Grounding} \\
\text{Proof} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{Causal Explanation} \arrow[r] & \text{Causality} \\
\text{Conceptual explanation} \arrow[r] \arrow[d] & \text{Grounding} \\
\text{Proof} \arrow[dr] & \\
 & \text{Derivation}
\end{tikzcd}
$$

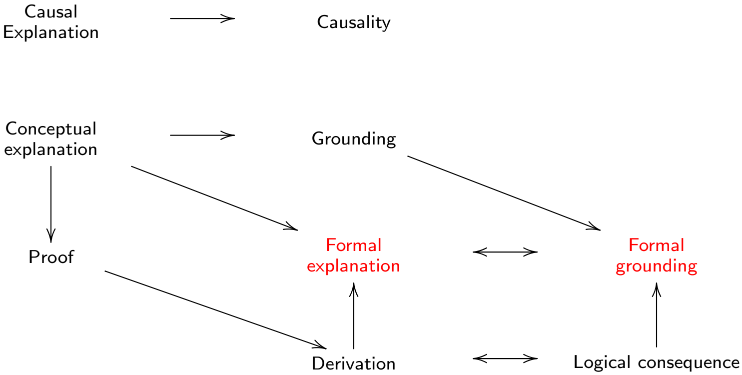
$$
\begin{tikzcd}
\text{Causal Explanation} \arrow[r] & \text{Causality} \\
\text{Conceptual explanation} \arrow[r] \arrow[d] \arrow[dr] & \text{Grounding} \\
\text{Proof} \arrow[dr] & \textcolor{red}{\text{Formal explanation}} \\
 & \text{Derivation} \arrow[u]
\end{tikzcd}
$$


Causal Explanation
Causality
Conceptual explanation
Grounding
Proof
Formal explanation
Derivation


Causal
Explanation
Causality
Conceptual
explanation
Grounding
Proof
Formal
explanation
Derivation
Logical consequence

$$
\begin{tikzcd}
\text{Causal Explanation} \arrow[r] & \text{Causality} & \\
\text{Conceptual explanation} \arrow[r] \arrow[d] \arrow[dr] & \textcolor{red}{\text{Grounding}} & \\
\text{Proof} \arrow[dr] & \text{Formal explanation} & \\
 & \text{Derivation} \arrow[u] \arrow[r, leftrightarrow] & \text{Logical consequence}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{Causal Explanation} \arrow[r] & \text{Causality} & \\
\text{Conceptual explanation} \arrow[r] \arrow[d] \arrow[dr] & \text{Grounding} \arrow[dr] & \\
\text{Proof} \arrow[dr] & \text{Formal explanation} & \text{Formal grounding} \\
& \text{Derivation} \arrow[u] \arrow[r, leftrightarrow] & \text{Logical consequence} \arrow[u]
\end{tikzcd}
$$


Causal Explanation
Causality
Conceptual explanation
Grounding
Proof
Formal explanation
Formal grounding
Derivation
Logical consequence

Notation

Notation

Formal explanation

$\Vdash$

Notation

Formal explanation

$\Vdash$

Formal grounding

$\Vvdash$

Notation

For any multiset of formulas of *FOL* M, and any formula A

Formal explanation

$M \Vdash A$

Formal grounding

$M \Vvdash A$

$M \Vdash A$ $\qquad$ $M \Vvdash A$

Total

$$M \Vdash A \qquad\qquad M \models\!\!\!\mid A$$

Total formal explanation/total grounding relation.

Reasons/conditions

$$M' \mid M \Vdash A \qquad\qquad M' \mid M \models A$$

Formal explanation/formal grounding carry a distinction between reasons and conditions.

Billy is my brother and Suzy is my sister. I have a nephew. The reason why I have a nephew is that my sister has a child. Indeed a nephew is the son of my brother or my sister and my sister (Suzy) has a child. My brother could have had a child, but he does not. Hence my brother having a child is merely a potential reason of why I have a nephew.[1]

[1] There is a stringent parallel with causation, see Menzies and Beebee (2020).

My sister (Suzy) has a child

My brother (Billy) has a child

My sister (Suzy) has a child

My brother (Billy) has a child

I have a nephew

My sister (Suzy) has a child

My brother (Billy) has a child

I have a nephew

My sister (Suzy)
has a child

Under the condition
that my brother does
not have a child

I have nephew

$M' \mid M \Vdash A$ $\qquad$ $M' \mid M \models A$

$M' \mid M \Vdash A$ $\qquad$ $M' \mid M \models A$

Logical analysis of formal grounding

$$M' \mid M \Vdash A$$

$$M' \mid M \Vdash A$$

What makes a consequence relation an explanatory one?[2]

[2]See F. Poggiolesi, On defining the notion of complete and immediate grounding, Synthese (2016). F. Poggiolesi and N. Francez, Towards a generalization of the logic of grounding, Theoria (2020). F. Genco, Formal Explanations as Logical Derivations (Francesco A. Genco). Journal of Applied Non-Classical Logics (2021).

$$M' \mid M \Vdash A$$

Which features define a grounding relation?

$$M' \mid M \Vdash A$$

Dependence.

$$M' \mid M \Vdash A$$

$$M' \mid F_1, ..., F_n \Vdash A$$

$$E_1, ..., E_m \mid F_1, ..., F_n \Vdash A$$

$$F_1, \dots F_n \models A$$

Not only is the conclusion a consequence of the ground(s).

$$\neg F_1, \ldots F_n \models A$$

If the premisses were somehow changed

$$F_1, \ldots \neg F_n \models A$$

If the premisses were somehow changed

$$\neg F_1, \ldots \neg F_n \models A$$

If the premisses were somehow changed

$$\neg E_1, \ldots \neg E_m, \neg F_1, \ldots \neg F_n \models A$$

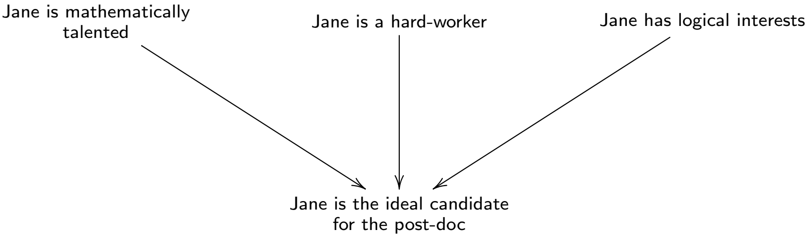
If the premisses were somehow changed plus the conditions,

$$\neg E_1, \ldots \neg E_m, \neg F_1, \ldots \neg F_n \models \neg A$$

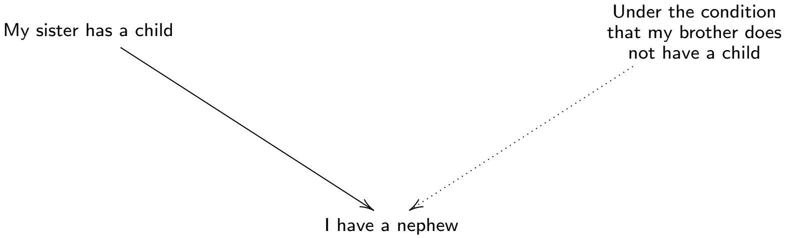
If the premisses were somehow changed plus the conditions, the change would affect the conclusion.

$$\neg E_1, \ldots \neg E_m, \neg F_1, \ldots \neg F_n \models \neg A$$

From the negation of some (even all) grounds + conditions, the negation of the conclusion follows.


Jane is mathematically talented
Jane is a hard-worker
Jane has logical interests
Jane is the ideal candidate for the post-doc

$$\begin{tikzcd}
\text{Jane is not mathematically talented} \arrow[dr] & \text{Jane is a hard-worker} \arrow[d] & \text{Jane has logical interests} \arrow[dl] \\
& \text{Jane is not the ideal candidate for the post-doc} &
\end{tikzcd}$$


My sister has a child
Under the condition that my brother does not have a child
I have a nephew

My sister does
not have a child

Under the condition
that my brother does
not have a child

I do not have nephew

$$M' \mid M \Vdash A$$

$$M' \mid M \Vdash A$$

A grounding relation is a dependence relation between premisses and the conclusion.

.

$$M' \mid M \Vdash A$$

Is that it?

.

$$M' \mid M \Vdash A$$

Explanatory relations are notoriously asymmetric relations.

.

$$F \Vvdash A$$

In case of a unique ground the dependence boils down to an equivalence.

.

$$F \Leftrightarrow A$$

In case of a unique ground the dependence boils down to an equivalence.

.

For any two points x and y, there always exists a third z such that $l(xy) = l(yz) = l(xz)$.

$\Longleftrightarrow$

For any two circles X and Y, one with center in x and radius xy, the other with center in y and radius xy, there exists a point z where they intersect and which is such that $l(xy) = l(yz) = l(xz)$.

We need an ingredient which establishes what explains what.

.

$$M' \mid M \Vdash A$$

According to a long philosophical tradition, the ingredient is complexity!

.

$$M' \mid M \Vdash A$$

The simplest premisses ground the more complex conclusion.

$$M' \mid M \Vdash A$$

.

Moreover, the complexity's increase from the grounds to their conclusion should be of a particular type: the formulas by means of which a sentence is grounded should correspond to a decomposition of the sentence itself.

$$M' \mid M \Vdash A$$

.

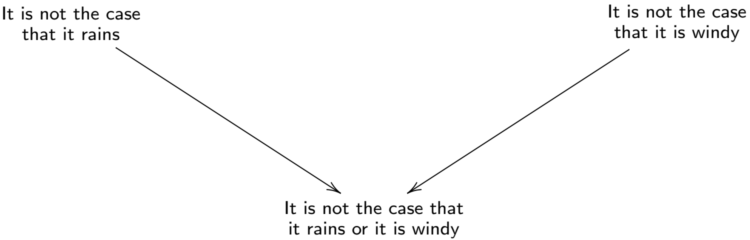
At first complexity seem to correspond to logical complexity and related subformula.

$$M' \mid M \Vdash A$$

.

Unfortunately, the formal grasp of the notion of explanation trough the notion of subformula turns out to be defective in several respects. (Adaptation of) Khale and Pulcini (2014).

Counterexample 1



Counterexample 1

$$\begin{tikzcd} \neg p \arrow[dr] & & \neg q \arrow[dl] \\ & \neg(p \vee q) & \end{tikzcd}$$

Counterexample 2

x is a natural number if it is 0

x is a natural number if it is the successor of a natural number

x is a natural number if it is 0 or it is the successor of a natural number

Counterexample 2

$$\begin{tikzcd}
\forall x(Zx \rightarrow Nx) \arrow[dr] & & \forall x(SNx \rightarrow Nx) \arrow[dl] \\
& \forall x((Zx \vee SNx) \rightarrow Nx) &
\end{tikzcd}$$

Counterexample 2

$$\begin{tikzcd}[column sep=small]
\forall x(Zx \rightarrow Nx) \arrow[dr] & & \forall x(SNx \rightarrow Nx) \arrow[dl] \\
& \forall x((Zx \vee SNx) \rightarrow Nx) &
\end{tikzcd}$$

See *Deep inferences*, http://alessio.guglielmi.name/res/cos/.

We will enrich the notion of complexity/subformula so for them to fit in an explanatory framework.

We will consider the well-formed formulas of the language of first-order logic and divide them in a new hierarchy of complexity, g-complexity, which extends the standard one by taking into account 1.

In accordance with the new notion of g-complexity, we will define another relation of subformula, g-subformula, that extends the standard one by taking into account 2.

Level 0

Level 0 Pc, Qc, Rc, ...

Level 0 $\quad Pc$, Qc, Rc, ..., $\neg Pc$, $\neg Qc$, $\neg Rc$, ...

Level 1

Level 0 Pc, Qc, Rc, ..., $\neg Pc$, $\neg Qc$, $\neg Rc$, ...

Level 1 $Pc \wedge Qc,$

Level 0 $Pc,\ Qc,\ Rc,\ \ldots,\ \neg Pc,\ \neg Qc,\ \neg Rc,\ \ldots.$

Level 1 $Pc \vee Qc,$

Level 0 $Pc,\ Qc,\ Rc,\ ...,\ \neg Pc,\ \neg Qc,\ \neg Rc,\ ....$

Level 1 $Pc \circ Qc$,

Level 0 Pc, Qc, Rc, ..., $\neg Pc$, $\neg Qc$, $\neg Rc$, ...

Level 1 $\qquad Pc \circ Qc, \qquad \forall x Px,$

Level 0 $\qquad Pc,\ Qc,\ Rc,\ ...,\ \neg Pc,\ \neg Qc,\ \neg Rc,\ ....$

Level 1 $\quad Pc \circ Qc, \quad \exists x Px,$

Level 0 $\quad Pc,\ Qc,\ Rc,\ ...,\ \neg Pc,\ \neg Qc,\ \neg Rc,\ ....$

Level 1 $Pc \circ Qc$, $\odot x Px$,

Level 0 Pc, Qc, Rc, ..., $\neg Pc$, $\neg Qc$, $\neg Rc$, ...

Level 1 $\quad Pc \circ Qc, \neg(Pc \circ Qc), \odot x Px,$

Level 0 $\quad Pc, Qc, Rc, ..., \neg Pc, \neg Qc, \neg Rc, ...$

Level 1 $\quad Pc \circ Qc, \neg(Pc \circ Qc), \odot x Px, \neg \odot x Px,$

Level 0 $\quad Pc, Qc, Rc, ..., \neg Pc, \neg Qc, \neg Rc, ...$

$\neg\neg\neg Pc$, ...

Level 1 $\neg\neg Pc$, ..., $Pc \circ Qc$, $\neg(Pc \circ Qc)$, $\odot x Px$, $\neg \odot x Px$, $\ldots$

Level 0 Pc, Qc, Rc, ..., $\neg Pc$, $\neg Qc$, $\neg Rc$, ...

$\vdots$

Level 2

$\neg\neg\neg Pc$, ...

Level 1 $\neg\neg Pc$, ..., $Pc \circ Qc$, $\neg(Pc \circ Qc)$, $\odot x Px$, $\neg \odot x Px$, $\ldots$

Level 0 Pc, Qc, Rc, ..., $\neg Pc$, $\neg Qc$, $\neg Rc$, ...

$\vdots$

$\neg\neg\neg\neg\neg\neg Pc$

Level 2 $\quad \neg\neg\neg\neg Pc$, $\neg\neg(Pc \circ Qc)$, $\forall y \forall x(P(x, y))$, $Rc \vee (Pc \wedge Qc)$

$\neg\neg\neg Pc$, ...

Level 1 $\quad \neg\neg Pc$, ..., $Pc \circ Qc$, $\neg(Pc \circ Qc)$, $\odot x Px$, $\neg \odot x Px$, $\ldots$

Level 0 $\quad Pc$, Qc, Rc, ..., $\neg Pc$, $\neg Qc$, $\neg Rc$, ...

$\vdots$

$\neg\neg\neg\neg\neg\neg Pc$

Level 2 $\quad \neg\neg\neg\neg Pc$, $\neg\neg(Pc \circ Qc)$, $\forall y \forall x(P(x, y))$, $Rc \vee (Pc \wedge Qc)$

$\neg\neg\neg Pc$, ...

Level 1 $\quad \neg\neg Pc$, ..., $Pc \circ Qc$, $\neg(Pc \circ Qc)$, $\odot x Px$, $\neg \odot x Px$, ...

Level 0 $\quad Pc$, Qc, Rc, ..., $\neg Pc$, $\neg Qc$, $\neg Rc$, ...

$\vdots$

$\neg\neg\neg\neg\neg\neg Pc$

Level 2 $\neg\neg\neg\neg\neg Pc$, $\neg\neg(Pc \circ Qc)$, $\forall y \forall x(P(x,y))$, $Rc \vee (Pc \wedge Qc)$

$\neg\neg\neg Pc$, ...

Level 1 $\neg\neg Pc$, ..., $Pc \circ Qc$, $\neg(Pc \circ Qc)$, $\odot x Px$, $\neg \odot x Px$, ...

Level 0 Pc, Qc, Rc, ..., $\neg Pc$, $\neg Qc$, $\neg Rc$, ...

$\vdots$

$\neg\neg\neg\neg\neg\neg Pc$

Level 2 $\neg\neg\neg\neg Pc$, $\neg\neg(Pc \circ Qc)$, $\forall y \forall x(P(x,y))$, $Rc \vee (Qc \wedge Pc)$

$\neg\neg\neg Pc$, ...

Level 1 $\neg\neg Pc$, ..., $Pc \circ Qc$, $\neg(Pc \circ Qc)$, $\odot x Px$, $\neg \odot x Px$, $\ldots$

Level 0 Pc, Qc, Rc, ..., $\neg Pc$, $\neg Qc$, $\neg Rc$, ...

$\vdots$

$\neg\neg\neg\neg\neg\neg Pc$

Level 2 $\quad \neg\neg\neg\neg\neg Pc,\ \neg\neg(Pc \circ Qc),\ \forall y \forall x(P(x,y)),\ Rc \vee (Qc \wedge Pc)$

$\neg\neg\neg Pc$, ...

Level 1 $\quad \neg\neg Pc,$ ..., $Pc \circ Qc,\ \neg(Pc \circ Qc),\ \odot x Px,\ \neg \odot x Px, \ldots$

Level 0 $\quad Pc,\ Qc,\ Rc,$ ..., $\neg Pc,\ \neg Qc,\ \neg Rc,$ ....

$\vdots$

$\neg\neg\neg\neg\neg\neg Pc$

Level 2 $\neg\neg\neg\neg Pc$, $\neg\neg(Pc \circ Qc)$, $\forall x \forall y(P(x,y))$, $Rc \vee (Qc \wedge Pc)$

$\neg\neg\neg Pc$, ...

Level 1 $\neg\neg Pc$, ..., $Pc \circ Qc$, $\neg(Pc \circ Qc)$, $\odot x Px$, $\neg \odot x Px$, $\ldots$

Level 0 Pc, Qc, Rc, ..., $\neg Pc$, $\neg Qc$, $\neg Rc$, ...

$\vdots$

$\neg\neg\neg\neg\neg\neg Pc$

Level 2 $\neg\neg\neg\neg\neg Pc$, $\neg\neg(Pc \circ Qc)$, $\forall k \forall w(P(k,w))$, $Rc \vee (Qc \wedge Pc)$

$\neg\neg\neg Pc$, ...

Level 1 $\neg\neg Pc$, ..., $Pc \circ Qc$, $\neg(Pc \circ Qc)$, $\odot x Px$, $\neg \odot x Px$, ...

Level 0 Pc, Qc, Rc, ..., $\neg Pc$, $\neg Qc$, $\neg Rc$, ...

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \to Qx \wedge Ry)$$

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx \wedge Ry)$$

What are its (immediate) subformulas?

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx \wedge Ry)$$

It depends on which part of the formula we focus on.

$$\exists(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx \wedge Ry)$$

It depends on which part of the formula we focus on.

$$\exists(Sx \land Tx) \lor \forall x \forall y(Px \to Qx \land Ry)$$

$\exists x(Sx \land Tx)$ $\forall x \forall y(Px \to Qx \land Ry)$

$$\exists(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx \wedge Ry)$$

$\exists x(Sx \wedge Tx)$ $\forall x \forall y(Px \rightarrow Qx \wedge Ry)$

$\neg\exists x(Sx \wedge Tx)$ $\neg\forall x \forall y(Px \rightarrow Qx \wedge Ry)$

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx \wedge Ry)$$

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx \wedge Ry)$$

Now suppose we focus on another part of the formula.

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx \wedge Ry)$$

Now suppose we focus on another part of the formula.

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx \wedge Ry)$$

A[]

$$\exists x(Sx \land Tx) \lor \forall x \forall y(Px \to Qx \land Ry)$$

$$\mathsf{A}[Qx \land Ry]$$

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx \wedge Ry)$$

$A[Qx \wedge Ry]$

$A[Qx]$

$A[Ry]$

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx \wedge Ry)$$

$$A[Qx \wedge Ry]$$

$$A[Qx]$$

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx)$$

$$A[Ry]$$

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Ry)$$

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx \wedge Ry)$$

$$A[Qx \wedge Ry]$$

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx \wedge Ry)$$

$$A[Qx \wedge Ry]$$

$\mathrm{A}[\neg Qx]$

$\mathrm{A}[\neg Ry]$

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx \wedge Ry)$$

$$A[Qx \wedge Ry]$$

$$\mathrm{A}[\neg Qx]$$

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow \neg Qx)$$

$$\mathrm{A}[\neg Ry]$$

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow \neg Ry)$$

$$\exists x(Sx \land Tx) \lor \forall x \forall y(Px \to Qx \land Ry)$$

$$A[Qx \land Ry]$$

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \to Qx \wedge Ry)$$

$$A[Qx \wedge Ry]$$

$$\neg A[Qx]$$

$$\neg A[Ry]$$

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx \wedge Ry)$$

$$A[Qx \wedge Ry]$$

$$\neg A[Qx]$$

$$\neg \exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx)$$

$$\neg A[Ry]$$

$$\neg \exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Ry)$$

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx \wedge Ry)$$

$$A[Qx \wedge Ry]$$

$$\exists x(Sx \land Tx) \lor \forall x \forall y(Px \to Qx \land Ry)$$

$$A[Qx \land Ry]$$

$$\neg A[\neg Qx] \qquad \neg A[\neg Ry]$$

$$\exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx \wedge Ry)$$

$$A[Qx \wedge Ry]$$

$$\neg A[\neg Qx]$$

$$\neg \exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow \neg Qx)$$

$$\neg A[\neg Ry]$$

$$\neg \exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow \neg Ry)$$

$$M' \mid M \Vdash A$$

$$M' \mid M \Vdash A$$

A grounding relation is a dependence relation where the grounds and conditions are g-subformulas of the conclusion.

For a more precise definition...

Definition

If $F[.]$ is a context, the *scope of a context*, $SC(F)$ and the inverse scope $SC^{inv}(F)$, are defined inductively in the following way:

- if $F[.] = [.]$ or $F[.] \neq \overbrace{\neg \ldots \neg}^{2n}[.]$ for $n \geqslant 0$ then $SC(F) = SC^{inv}(F) = \emptyset$,
- if $F[.] = G \circ E[.]$ or $E[.] \circ G$, then $SC(F) = SC(E)$ and $SC^{inv}(F) = SC^{inv}(E)$,
- if $F[.] = \forall x E[.]$, then $SC(F) = \forall x.(SC(E))$ and $SC^{inv}(F) = \exists x.SC^{inv}(E)$
- if $F[.] = \exists x E[.]$, then $SC(F) = \exists x.(SC(E))$ and $SC^{inv}(F) = \forall x.SC^{inv}(E)$
- if $F[.] = \neg E[.]$, then $SC(F) = SC^{inv}(E)$ and $SC^{inv}(F) = SC(E)$.

Definition

Let F be a formula of $\mathcal{L}$, then the free variables of F, $FV(F)$, are standardly defined as those variables occurring in F which are not bound by any quantifier. We define the *restricted free variables* of a formula F, $FV^+(F)$, in the following way

- if F is not of the form $G \circ G'$, then $FV^+(F) = FV(F)$,
- if F is of the form $G \circ G'$, then $FV^+(F) = FV(G) \cap FV(G')$.

Definition

Let $C[F]$ be a formula in a context, and let $FV^+(F) = x_1, ..., x_n$. The scope of a context $C[.]$ relative to the formula F, $SC_F(C)$, is the result of removing from $SC(C)$ any quantifier that is not of the form $\odot x_1, ..., \odot x_n$.

Definition

For any finite multisets of $\mathcal{CF}$ $M = \{A_1[D_1], ..., A_m[D_m]\}$ and $N = \{A'_1[C_1], ..., A'_n[C_n]\}$(which could be empty), and for any $\mathcal{CF}$ $F[B]$, under the condition that $N^{\perp}$, M is a total and immediate formal ground of $F[B]$, in symbols $N \mid M \Vdash F[B]$, if, and only if, for any E such that $SC(E) = SC(F)$ and $E \in \mathcal{P}$ if, and only if, $F \in \mathcal{P}$, we have:

1. $E[D_1], ..., E[D_m] \models E[B]$,
2. for some non empty (possibly non proper) submultiset M' of M, such that $M' = \{A_{k1}[D_{k1}], ..., A_{kr}[D_{kr}]\}$, we have that $(E[C_1])^{\perp}, ..., (E[C_n])^{\perp}, (E[D_{k1}])^{\perp}, ..., (E[D_{kr}])^{\perp}, M^{-}/E \models (E[B])^{\perp}$.
3. $N \cup M$ is a multiset of immediate and distinguished g-subformulas of $F[B]$.

where $M^{-} = M - M'$ and $M^{-}/E = \{E[D_z] \mid A_z[D_z] \in M^{-}\}$.

Example 1

It is not the case
that it rains

It is not the case
that it is windy

It is not the case that
it rains or it is windy

Example 1

$$\begin{tikzcd} \neg p \arrow[dr] & & \neg q \arrow[dl] \\ & \neg(p \vee q) & \end{tikzcd}$$

Example 1

$$\begin{tikzcd} \neg p \arrow[dr] & & \neg q \arrow[dl] \\ & \neg(p \vee q) & \end{tikzcd}$$

$\neg p, \neg q \models \neg(p \vee q)$

Example 1

$$\begin{tikzcd} p \arrow[dr] & & \neg q \arrow[dl] \\ & p \vee q & \end{tikzcd}$$

$p, \neg q \models (p \vee q)$

Example 1

$$\begin{tikzcd}
\neg p \arrow[dr] & & \neg q \arrow[dl] \\
& \neg(p \vee q) &
\end{tikzcd}$$

$\neg p$ and $\neg q$ are g-subformulas of $\neg(p \vee q)$

Example 2

x is a natural number if it is 0

x is a natural number if it is the successor of a natural number

x is a natural number if it is 0 or it is the successor of a natural number

Example 2

$$
\begin{tikzcd}
\forall x(Zx \rightarrow Nx) \arrow[dr] & & \forall x(SNx \rightarrow Nx) \arrow[dl] \\
& \forall x((Zx \vee SNx) \rightarrow Nx) &
\end{tikzcd}
$$

Example 2

$$\begin{tikzcd}
\forall x(Zx \to Nx) \arrow[dr] & & \forall x(SNx \to Nx) \arrow[dl] \\
& \forall x((Zx \lor SNx) \to Nx) &
\end{tikzcd}$$

$\forall x(Zx \to Nx), \forall x(SNx \to Nx) \models \forall x(Zx \lor SNx \to Nx)$

Example 2

$$\begin{tikzcd}
\neg(\forall x(Zx \to Nx)) \arrow[dr] & & \forall x(SNx \to Nx) \arrow[dl] \\
& \neg(\forall x((Zx \vee SNx) \to Nx)) &
\end{tikzcd}$$

$\neg(\forall x(Zx \to Nx)), \forall x(SNx \to Nx) \models \neg(\forall x(Zx \vee SNx \to Nx))$

Example 2

$$\begin{tikzcd}
\forall x(Zx \to Nx) \arrow[dr] & & \forall x(SNx \to Nx) \arrow[dl] \\
& \forall x((Zx \lor SNx) \to Nx) &
\end{tikzcd}$$

$\forall x(Zx \to Nx)$ and $\forall x(SNx \to Nx)$ are g-subformulas of $\forall x(Zx \lor SNx \to Nx)$

Logical analysis of formal explanation

$$M' \mid M \Vdash A$$

$$M' \mid M \Vdash A$$

$$M' \mid M \Vdash A$$

We need to introduce (explanatory) rules that define a formal explanation.

$$M' \mid M \Vdash A$$

The rules will be such that their premisses are the total grounds of the conclusion.

$$M' \mid M \Vdash A$$

Hence they will reflect at the proof-theoretical level the features of the grounding relation.

We will work with the sequent calculus extending previous results obtained in natural deduction.[2]

[2]See Poggiolesi, F., *On constructing a logic for the notion of complete and immediate formal grounding*, Synthese, 2018. Genco, F. *What Stands Between Grounding Rules and Logical Rules Is the Excluded Middle*, RSL, forthcoming.

Gfcl

Initial Sequents $p, M \Rightarrow M, p$

Structural Rule

$$\frac{M \Rightarrow N}{P \Rightarrow Q \mid M \Rightarrow N}\ WC$$

Propositional Logical Rules

$$\frac{M \Rightarrow N, A}{\neg A, M \Rightarrow N}\ \neg L \qquad \frac{A, M \Rightarrow N}{M \Rightarrow N, \neg A}\ \neg R$$

$$\frac{A_0, A_1, M \Rightarrow N}{A_0 \wedge A_1, M \Rightarrow N}\ \wedge L \qquad \frac{M \Rightarrow N, A \qquad M \Rightarrow N, B}{M \Rightarrow N, A \wedge B}\ \wedge R$$

First-order Logical Rules

$$\frac{\forall x A, A(x/t), M \Rightarrow N}{\forall x A, M \Rightarrow N}\ \forall L \qquad \frac{M \Rightarrow N, A(y)}{M \Rightarrow N, \forall x A}\ \forall R$$

where the y does not occur neither in M nor in N.

We now add explanatory rules.

$$\begin{array}{c} \vdots \qquad \vdots \\ \hline\hline \\ \vdots \end{array}$$

One premise

$$\begin{array}{c}\vdots\\ = \\ \\ \vdots\end{array}$$

Two premises

$$\begin{array}{c} \vdots \quad \vdots \\ \hline\hline \\ \vdots \end{array}$$

One premise + one condition

$$
\begin{array}{cc}
\vdots \mid & \vdots \\
\hline\hline
\multicolumn{2}{c}{\vdots}
\end{array}
$$

One premise + one condition

$$\genfrac{}{}{}{}{\vdots \mid \quad \vdots}{\vdots} \qquad\qquad\qquad \ldots \mid \ldots \Vdash \ldots$$

Definition

Formulas in contexts, as C in $A[C]$, occur with either a positive or a negative polarity, where polarities are defined in a standard way, e.g. see Troelstra and Schwichtenberg (1996).

Definition

The *converse of a formula* A, written $A^{\perp}$, is defined as follows:

$$A^{\perp} = \begin{cases} \neg^{n-1}E, & \textit{if } A = \neg^{n}E \textit{ and } \mathrm{n} \textit{ is odd} \\ \neg^{n+1}E, & \textit{if } A = \neg^{n}E \textit{ and } \mathrm{n} \textit{ is even} \end{cases}$$

where the main connective in E is not a negation, $n \geqslant 0$ and 0 is taken to be an even number.

Double negation

$$\frac{M \Rightarrow N, A[B]}{M \Rightarrow N, A[\neg\neg B]} \; \neg\neg$$

Double negation

$$\frac{M \Rightarrow N, B}{M \Rightarrow N, \neg\neg B}\ \neg\neg$$

Double negation

$$\frac{M \Rightarrow N, A[B]}{M \Rightarrow N, A[\neg\neg B]} \; \neg\neg$$

Conjunction/Disjunction

$$\frac{M \Rightarrow N, A[B] \quad M \Rightarrow N, A[C]}{M \Rightarrow N, A[B \circ C]} \circ_1 \qquad \frac{M \Rightarrow N, A[B_j] \mid \quad M \Rightarrow N, A[B_i]}{M \Rightarrow N, A[B_1 \circ B_2]} \circ_2$$

Conjunction/Disjunction

$$\frac{M \Rightarrow N, A[B] \quad M \Rightarrow N, A[C]}{M \Rightarrow N, A[B \circ C]} \circ_1 \qquad \frac{M \Rightarrow N, A[B_j] \mid \quad M \Rightarrow N, A[B_i]}{M \Rightarrow N, A[B_1 \circ B_2]} \circ_2$$

Rules obey certain restrictions - we do not list them all.

Conjunction/Disjunction

$$\frac{M \Rightarrow N, A[B] \quad M \Rightarrow N, A[C]}{M \Rightarrow N, A[B \wedge C]} \circ_1 \qquad \frac{M \Rightarrow N, A[B_j] \mid \quad M \Rightarrow N, A[B_i]}{M \Rightarrow N, A[B_1 \circ B_2]} \circ_2$$

If $A \in \mathcal{P}$ and $SC(A) = \emptyset$ or $SC(A) = \forall x_1, \ldots \forall x_n$.

Conjunction/Disjunction

$$\frac{M \Rightarrow N, A[B] \quad M \Rightarrow N, A[C]}{M \Rightarrow N, A[B \vee C]} \circ_1 \qquad \frac{M \Rightarrow N, A[B_j] \mid \quad M \Rightarrow N, A[B_i]}{M \Rightarrow N, A[B_1 \circ B_2]} \circ_2$$

If $A \in \mathcal{N}$ and $SC(A) = \emptyset$ or $SC(A) = \forall x_1, ...\forall x_n$.

Conjunction/Disjunction

$$\frac{M \Rightarrow N, A[B] \quad M \Rightarrow N, A[C]}{M \Rightarrow N, A[B \circ C]} \circ_1 \qquad \frac{M \Rightarrow N, A[B_j] \mid \quad M \Rightarrow N, A[B_i]}{M \Rightarrow N, A[B_1 \vee B_2]} \circ_2$$

If $A \in \mathcal{P}$ and $SC(A) = \emptyset$ or $SC(A) = \exists x_1, ... \exists x_n$.

Conjunction/Disjunction

$$\frac{M \Rightarrow N, A[B] \quad M \Rightarrow N, A[C]}{M \Rightarrow N, A[B \circ C]} \circ_1 \qquad \frac{M \Rightarrow N, A[B_j] \mid \quad M \Rightarrow N, A[B_i]}{M \Rightarrow N, A[B_1 \wedge B_2]} \circ_2$$

If $A \in \mathcal{N}$ and $SC(A) = \emptyset$ or $SC(A) = \exists x_1, ...\exists x_n$.

$$\dfrac{\Rightarrow \forall x(Zx \to Nx) \quad \Rightarrow \forall x(SNx \to Nx)}{\Rightarrow \forall x((Zx \vee SNx) \to Nx)} \circ 1$$

$$\cfrac{}{\Rightarrow \forall x(Nx \rightarrow (Ox \vee Ex))} \; NR$$

Negation of Conjunction/Disjunction

$$\frac{M \Rightarrow N, A[B^{\perp}] \quad M \Rightarrow N, A[C^{\perp}]}{M \Rightarrow N, A[\neg(B \circ C)]} \neg\circ_1 \qquad \frac{M \Rightarrow N, A[B_j^{\perp}] \mid \quad M \Rightarrow N, A[B_i^{\perp}]}{M \Rightarrow N, A[\neg(B_1 \circ B_2)]} \neg\circ_2$$

Negation of Conjunction/Disjunction

$$\frac{M \Rightarrow N, A[B^{\perp}] \quad M \Rightarrow N, A[C^{\perp}]}{M \Rightarrow N, A[\neg(B \circ C)]} \neg\circ_1 \qquad \frac{M \Rightarrow N, A[B_j^{\perp}] \mid \quad M \Rightarrow N, A[B_i^{\perp}]}{M \Rightarrow N, A[\neg(B_1 \circ B_2)]} \neg\circ_2$$

Rules continue to obey restrictions.

Negation of Conjunction/Disjunction

$$\frac{M \Rightarrow N, A[B^{\perp}] \quad M \Rightarrow N, A[C^{\perp}]}{M \Rightarrow N, A[\neg(B \vee C)]} \neg\circ_1 \qquad \frac{M \Rightarrow N, A[B_j^{\perp}] \mid \quad M \Rightarrow N, A[B_i^{\perp}]}{M \Rightarrow N, A[\neg(B_1 \circ B_2)]} \neg\circ_2$$

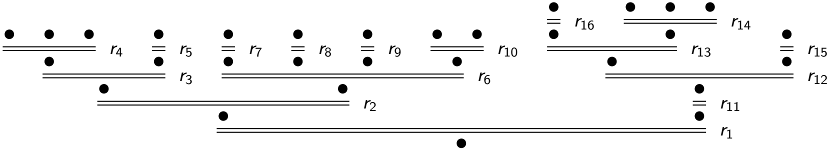
If $A \in \mathcal{P}$ and $SC(A) = \emptyset$ or $SC(A) = \forall x_1, ...\forall x_n$.

$$\dfrac{\Rightarrow p^{\perp} \quad \Rightarrow q^{\perp}}{\Rightarrow \neg(p \vee q)} \; \neg\circ 1$$

$$\dfrac{\Rightarrow \neg p \quad \Rightarrow \neg q}{\Rightarrow \neg(p \vee q)}\ \neg\circ 1$$

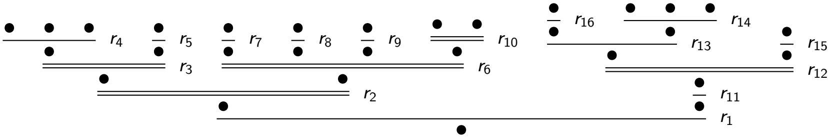
Thanks to the explanatory rules, we can now construct:

Thanks to the explanatory rules, we can now construct:



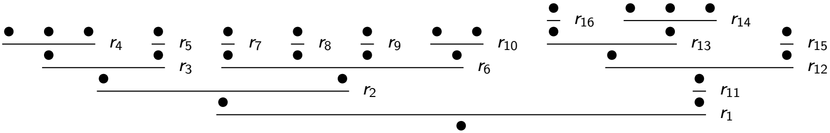
Formal explanations

Thanks to the explanatory rules, we can now construct:



Derivations with explanatory steps

Thanks to the explanatory rules, we can now construct:



Derivations

Definition

A derivation in **Gfcl** is a finite (upwardgrowing) tree with a single root. The nodes of the tree are labelled by sequents or sequents with a bar and the top nodes are labelled by initial sequents. For each non-terminal node, its label is connected with the labels of the immediate predecessor nodes according with one of the logical rules or one of the explanatory rules. The root of the tree is the conclusion of the whole derivation and its label is a theorem of the sequent calculus, in symbol $\vdash_{Gfcl} M \Rightarrow N$.

Definition

For any multiset of sequents S' (which might be empty), and for any multiset of sequents S, we say that under the condition $(S')^{\perp}$, there exists a complete and immediate formal explanation from S to $M \Rightarrow N$, in symbols $S' \mid S \Vdash M \Rightarrow N$ if, and only if, one of the explanatory rules $\neg\neg, \circ_1, \circ_2, \neg\circ_1, \neg\circ_2$ link S', S and $M \Rightarrow N$.

Definition

For any multiset of sequents S' (which might be empty), and for any multiset of sequents S, we say that under the condition $(S')^{\perp}$, S completely and mediately formally explain $M \Rightarrow N$, in symbols $S' \mid S \Vdash^m M \Rightarrow N$ if, and only if:

- $S' \mid S \Vdash M \Rightarrow N$,
- $S'' \mid S''' \Vdash M' \Rightarrow N'$ $S'''' \mid S''''', M' \Rightarrow N' \Vdash M \Rightarrow N$, and $S'' \cup S'''' = S'$ and $S''' \cup S''''' = S$.

Results/Directions of Future Research

Theorem (Soundness)

For any mutiset of sequents (possibly empty) S', S, and sequent $M \Rightarrow N$,

$$\textit{if } S' \mid S \Vdash M \Rightarrow N, \textit{ then } (S')^{\tau} \mid (S)^{\tau} \Vdash\!\!= \bigwedge M \rightarrow \bigvee N$$

where $(S')^{\tau}$, $(S)^{\tau}$ are the multiset of sequents standardly translated into formulas.

Theorem (Completeness)

For any multiset of closed formulas (possibly empty) N', N, and formula $A[C]$,

$$\textit{if } N' \mid N \Vdash A[C], \textit{ then } (N')^{\delta} \mid (N)^{\delta} \Vdash \Rightarrow A[C]$$

where for any $M = \{A[B_1], ...A[B_n]\}$, $(M)^{\delta} = \{\Rightarrow A[B_1], ..., \Rightarrow A[B_n]\}$.

Theorem (Admissibility)

Any explanatory rule is eliminable in **Gfcl**.

Application

We use formal explanations to properly modeling mathematical explanations.[a]

[a] See Mathematical explanations: an analysis via complexity and proof, *Philosophia Mathematica*, forthcoming. Also in a joint work with E. Pimentel.

Application

Formal explanations bear a strict connection with reasons and decisions in Explainable AI.[a]

[a] See *Decisions with reasons in sequent calculus style*, In preparation, joint work with B. Hill.

Directions of future research

- Formal explanation/formal grounding for logics different from classical first-order logic (intuitionistic logic, modal logic, ...).
- Explanatory rules and proof-theoretic semantics.[3]
- Explanatory rules and deep inferences.[4]
- Conceptual explanation/grounding and the links with causal explanation/causality.
- Semantics for this proof-theoretical apparoach.

[3] F. Poggiolesi, Grounding rules and (hyper-)isomorphic formulas, *Australasian Journal of Logic*, 17: 70-80, 2020.

[4] See http://alessio.guglielmi.name/res/cos/

THANK YOU!