Tame pairs of transseries fields

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Transseries

Let $\mathbb{T}$ be the differential field of logarithmic-exponential transseries constructed by Van den Dries–Macintyre–Marker. Example series in $\mathbb{T}$:

$$7e^{e^x+e^{x/2}+e^{x/4}+\dots} - 3e^{x^2} + 5x^{\sqrt{2}} - (\log x)^{\pi} + 42 + x^{-1} + x^{-2} + \cdots + e^{-x}$$

Remarks:

1. $\mathbb{T}$ is a real closed field equipped with a derivation.
2. $\mathbb{T}$ is exponentially bounded.
3. $\mathcal{O}_{\mathbb{T}} := \mathrm{conv}_{\mathbb{T}}(\mathbb{R}) = \{f \in \mathbb{T} : |f| \leqslant r \text{ for some } r \in \mathbb{R}^{>0}\}$ is a convex valuation ring of $\mathbb{T}$ distinguishing "bounded" and "large" elements.

Theorem (Aschenbrenner–Van den Dries–Van der Hoeven '17)

$(\mathbb{T}, \mathcal{O}_{\mathbb{T}})$ *is model complete.*

Nonstandard models

Let $K \succcurlyeq \mathbb{T}$ with $f > \mathbb{T}$ for some $f \in K$.

Example

1. Hyperseries (Bagayoko–Van der Hoeven–Kaplan '21+, B '22+)
2. surreal numbers (Berarducci–Mantova '18, ADH '19)
3. maximal Hardy fields (ADH '24+, AD '24+)

Let $\mathcal{O}_K \coloneqq \operatorname{conv}_K(\mathbb{R})$ and $\dot{\mathcal{O}} \coloneqq \operatorname{conv}_K(\mathbb{T})$.

Theorem (PC '24+)

$(K, \mathcal{O}_K, \dot{\mathcal{O}})$ *is model complete.*

Transserial tame pairs

Definition

A **transserial tame pair** is (K, L) such that:

1. $(K, \mathcal{O}_K), (L, \mathcal{O}_L) \models \mathrm{Th}(\mathbb{T})$, where $\mathcal{O}_K := \mathrm{conv}_K(C_K)$ and $C_K := \{c \in K : c' = 0\}$;
2. $L \subsetneq K$ is a proper differential subfield;
3. L is tame in K, i.e., $\dot{\mathcal{O}} = L + \dot{o}$, where $\dot{\mathcal{O}} := \mathrm{conv}_K(L)$ and $\dot{o} := \{a \in K : |a| < L^{>0}\}$.

Theorem (PC '24+)

1. *The theory of transserial tame pairs is complete and model complete.*
2. *Any subset of L^n definable in (K, L) is definable in L and any subset of C^n definable in (K, L) is definable in C.*

Case study: Hyperseries

BHK/B construct a differential field $\mathbb{H} \succcurlyeq \mathbb{T}$ containing $\exp_\alpha(x)$ and $\log_\alpha(x)$ for every ordinal α; e.g., $\exp_\omega(x)$ solves $E(x+1) = \exp E(x)$.

Let:

1. $\mathcal{O}_{\mathbb{H}} := \mathrm{conv}_{\mathbb{H}}(\mathbb{R})$ and $\dot{\mathcal{O}} := \mathrm{conv}_{\mathbb{H}}(\mathbb{T})$;
2. $\mathfrak{M}$ be the monomial group of $\mathbb{H}$ and $\mathfrak{B} := \mathfrak{M} \cap \dot{\mathcal{O}}^\times$;
3. $\mathbb{T}^* := \{f \in \mathbb{H} : \operatorname{supp} f \subseteq \mathfrak{B}\}$ and $\mathcal{O}_{\mathbb{T}^*} := \mathrm{conv}_{\mathbb{T}^*}(\mathbb{R})$.

Proposition

$(\mathbb{H}, \mathbb{T}^*)$ *is a transserial tame pair.*

Corollary

$(\mathbb{H}, \mathcal{O}_{\mathbb{H}}, \dot{\mathcal{O}})$ *and* $(\mathbb{H}, \dot{\mathcal{O}}, \mathbb{T}^*, \mathcal{O}_{\mathbb{T}^*})$ *are model complete.*

Thank you!