On the Definability and Complexity of Sets and Structures

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Definition (Classical mathematics)

Those mathematical subjects which, originally, have no connection to mathematical logic.

For example:

- Analysis
- Algebra
- Geometry
- ...

Goal

Use tools from mathematical logic to prove theorems from classical mathematics. In particular, find definable counterexamples.

A set $X \subset \mathbb{R}$ has the perfect set property if it is countable or if it contains a perfect (i.e. closed with no isolated points) subset.

(ZF)	every analytic set has PSP (Souslin)
(ZFC)	some set does not have PSP (Bernstein)
(ZF + AD)	every set has PSP (Mycielski, Swierczkowski)
(ZF + V = L)	some coanalytic set does not have PSP (Gödel)

Complexity (here Borel/projective hierarchy) informs about provability (over the "base theory" ZF)!

Fix some property of sets of reals P.

Question

For which sets is P provable? What is the "simplest" set failing P?

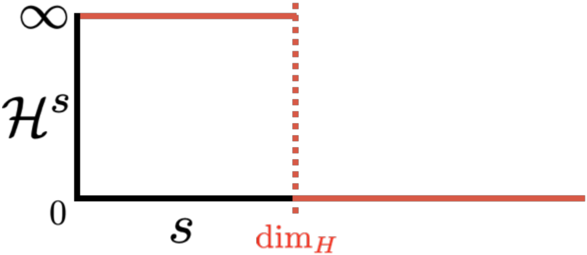
We use descriptive set theory.

$$\begin{array}{ccccc} & \mathbf{\Sigma}^1_1 & & \mathbf{\Sigma}^1_2 & \\ \mathbf{\Delta}^1_1 & & \mathbf{\Delta}^1_2 & & \cdots \\ & \mathbf{\Pi}^1_1 & & \mathbf{\Pi}^1_2 & \end{array}$$

Definition (Hausdorff dimension)

For $E \subset \mathbb{R}^n$

$$\dim_H(E) = \sup\{s \mid \mathcal{H}^s(E) = \infty\} = \inf\{s \mid \mathcal{H}^s(E) = 0\}.$$

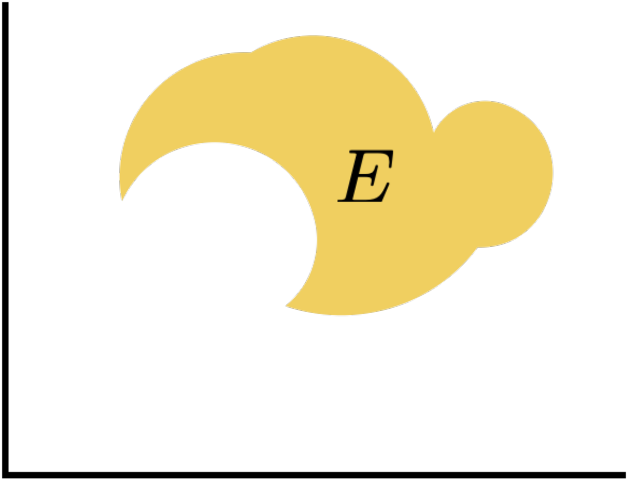


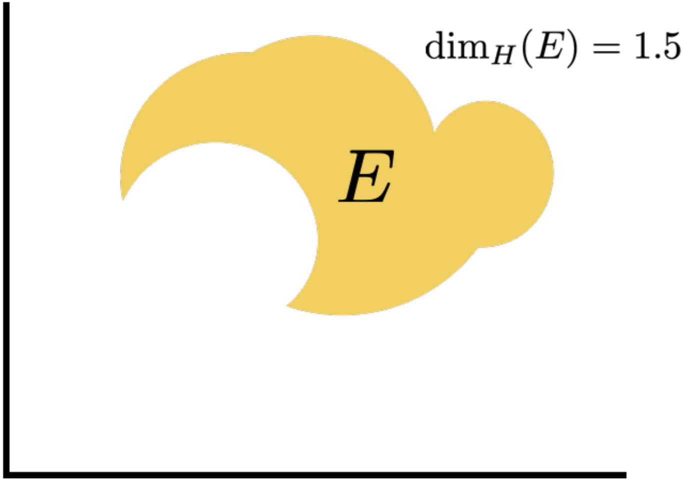
Marstrand's Projection Theorem (J. Marstrand, 1954)

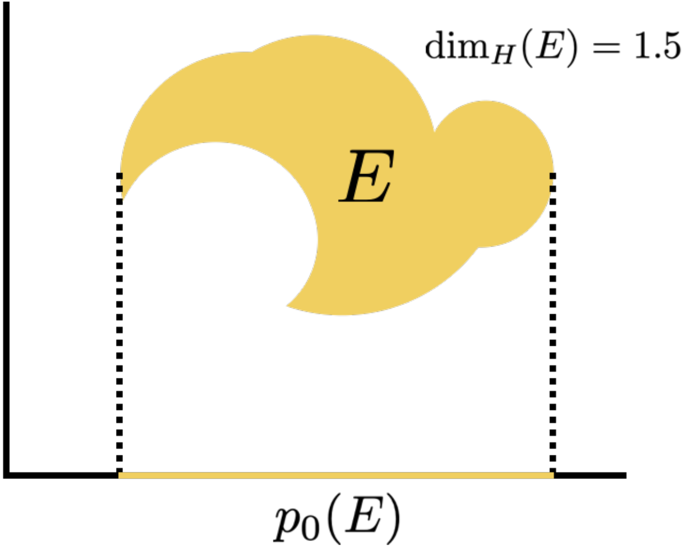
Let $E \subset \mathbb{R}^2$ be analytic. For almost all θ

$$\dim_H(p_\theta(E)) = \min\{\dim_H(E), 1\}$$

where p_θ is the orthogonal projection onto the line θ.


E


$\dim_H(E) = 1.5$
E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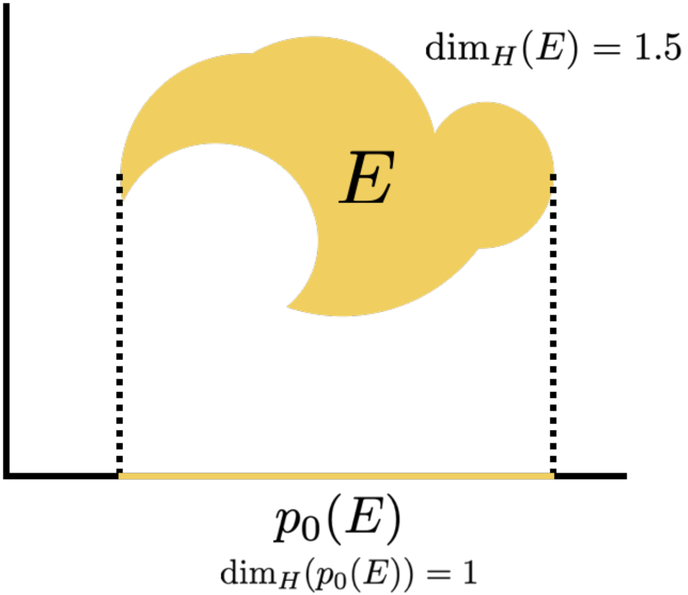
$p_0(E)$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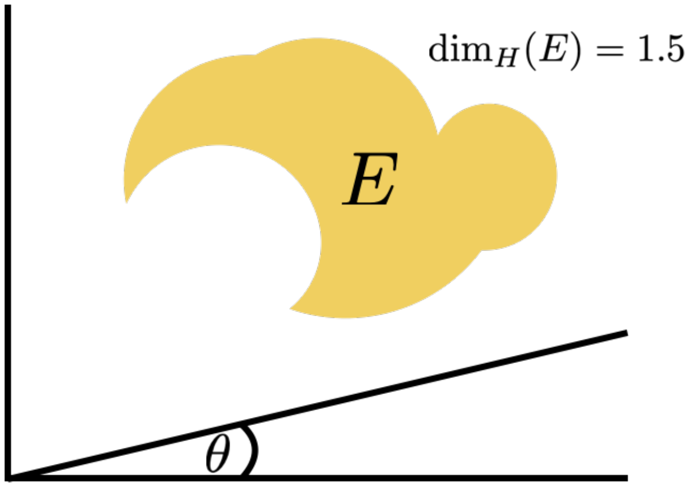
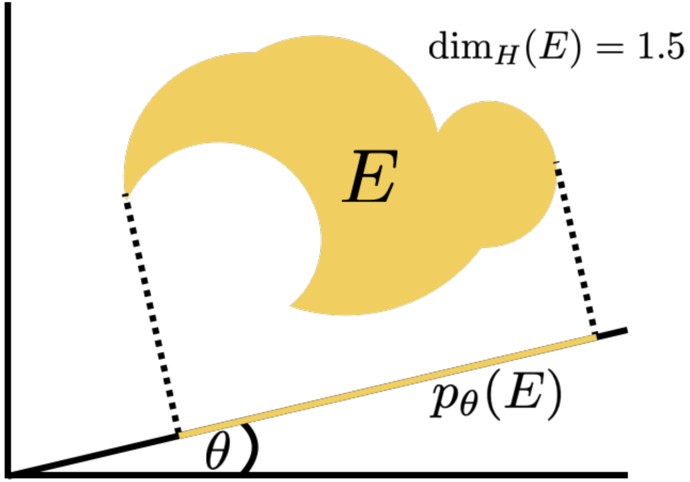
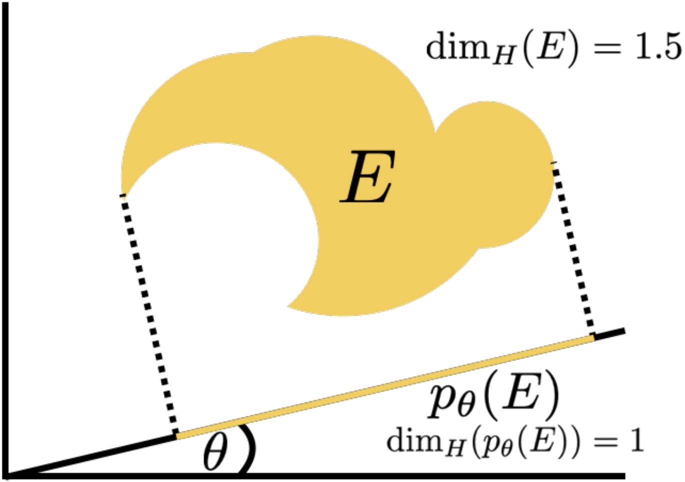


$\dim_H(E) = 1.5$
E
θ



$\dim_H(E) = 1.5$
E
$p_\theta(E)$
θ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E
$p_\theta(E)$
$\dim_H(p_\theta(E)) = 1$
θ

Does Marstrand's Theorem provably apply to more sets?

Theorem (Davies, 1979)

Assuming CH, *there exists a set* $E \subset \mathbb{R}^2$ *with* $\dim_H(E) = 1$ *while, for every* θ, *we have* $\dim_H(p_\theta(E)) = 0$.

A set $E \subset \mathbb{R}^2$ has the Marstrand property (MP) if Marstrand's theorem applies to it.

(ZF)	every analytic set has MP (Marstrand, Mattila)
$(\mathrm{ZFC} + \mathrm{CH})$	some set does not have MP (Davies)
$(\mathrm{ZF} + \mathrm{AD})$	?
$(\mathrm{ZF} + \mathrm{V} = \mathrm{L})$	?

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(ZF)	every analytic set has MP (Marstrand, Mattila)
(ZFC + CH)	some set does not have MP (Davies)
(ZF + AD)	?
(ZF + V = L)	some co-analytic set does not have MP (R.)

Marstrand's theorem (special case)

For every analytic $E \subset \mathbb{R}^2$ for which $\dim_H(E) = 1$ we have $\dim_H(p_\theta(E)) = 1$ for almost all θ.

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Theorem (R.)

($V{=}L$) *There exists a co-analytic* $E \subset \mathbb{R}^2$ *such that* $\dim_H(E) = 1$ *and* $\dim_H(p_\theta(E)) = 0$ *for all* θ.

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Theorem (R.)

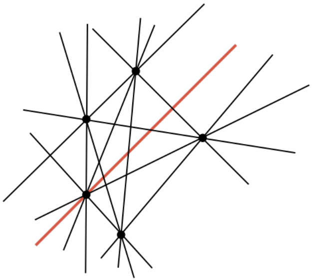
($V{=}L$) *For every* $\epsilon \in [0, 1)$ *there exists a co-analytic* $E_\epsilon \subset \mathbb{R}^2$ *such that* $\dim_H(E_\epsilon) = 1 + \epsilon$ *and* $\dim_H(p_\theta(E_\epsilon)) = \epsilon$ *for all* θ.

How does one construct such sets? By recursion!

Example (Two-point set)

(Sketch)

1. Take a listing of all lines using AC;
2. Build a partial solution;
3. Find a witness to extend the partial solution to the next line;
4. Continue until all lines are dealt with.



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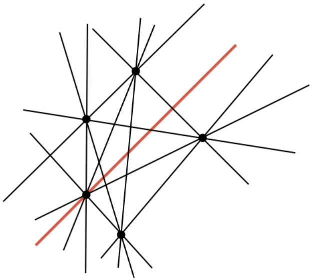

But sets constructed in this way are normally very complicated!

Theorem (Vidnyánszky)

(Informal statement, $\mathsf{V} = \mathsf{L}$*) If enough information can be coded into the witnesses during the recursion, then the resultant set is co-analytic.*

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Theorem (Lutz, Lutz)

The Hausdorff-dimension of a set of reals is determined by the (Kolmogorov) complexity of its points.

So $\dim_H$ is actually a local property!

Open questions

In fractal geometry:

- What about $\dim_H(E) < 1$?
- Packing dimension?

In set theory:

- What about the other extreme? Is it consistent that *every* set of reals satisfies Marstrand's theorem?

More generally:

- is there a general criterion that describes properties P which behave like the PSP?

Thank you