Hausdorff dimension and countable Borel equivalence relations

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joint work with Andrew Marks and Ted Slaman

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Countable Borel equivalence relations

Let X be a Polish space $(2^\omega, \omega^\omega, \mathbb{R}^n, ..)$. A Borel equivalence relation is an equivalence relation E s.t. $E \in \mathcal{B}(X \times X)$.
E is countable (CBER) if all its equivalence classes are countable.

Examples: - $\equiv_T$: $x \equiv_T y$ if x is computable from y and y is comp. from x

E_0: $x E_0 y$ $\exists n (\forall m > n)\ x(m) = y(m)$, $x, y \in 2^\omega$.

E_∞: shift action of the free group on 2-generators F_2.

For $x, y \in 2^{F_2}$ $x E_\infty y \Leftrightarrow \exists g \in F_2 \forall h\ x(gh) = y(h)$.

$id_\mathbb{R}$: $x \, id_\mathbb{R} \, y \Leftrightarrow x = y$.

For E, F equivalence relations on X, resp. Y, E is Borel reducible to F, $E \leq_B F$ if there is a Borel $f: X \to Y$ s.t. $x E y \Leftrightarrow f(x) F f(y)$.

$E \leq_B F$ intuitively tells you that from a definition of F one can get a definition of E.

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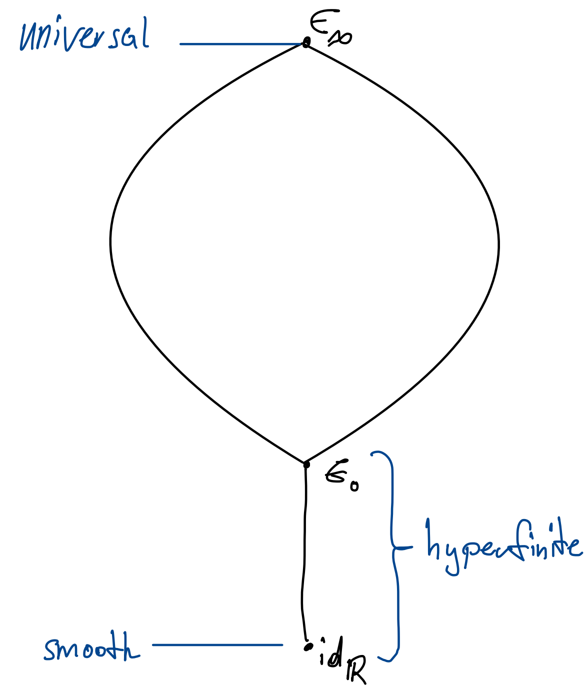

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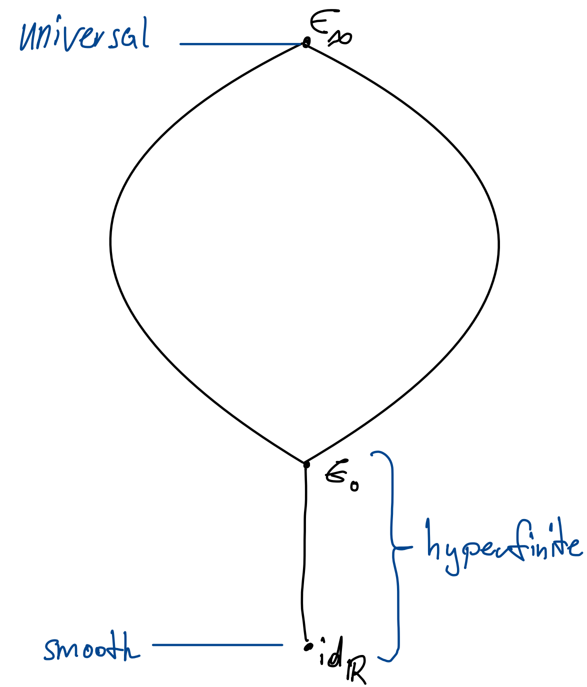
Q1. Is $\equiv_T$ universal? For neg. answer show $E_\infty \not\leq_B F$



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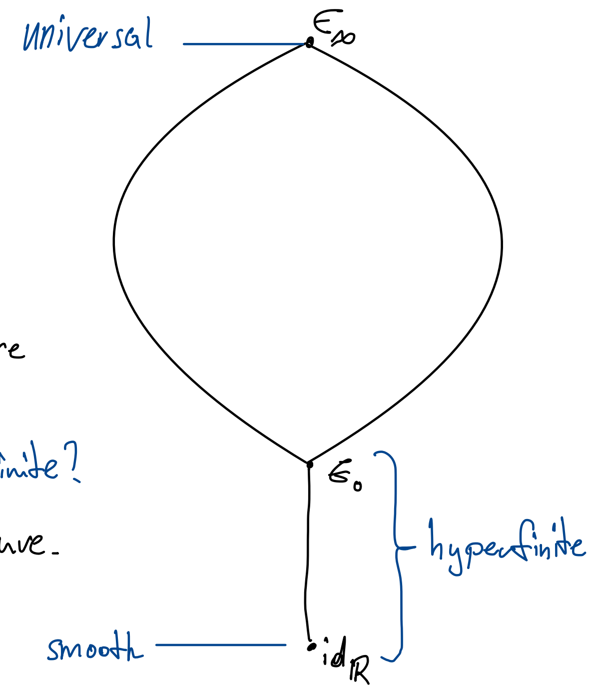
- Almost all non-trivial non-reducibility results use measure theory.

Q2. Is an increasing union of hyperfinite CBERs hyperfinite?

- true modulo a null set for any Borel probability measure.



Hausdorff and gauge measures

A gauge function $g:[0,\infty) \to [0,\infty)$ is an increasing fct that is right continuous, $g(0)=0$ and $g(t)>0$ if $t\neq 0$.

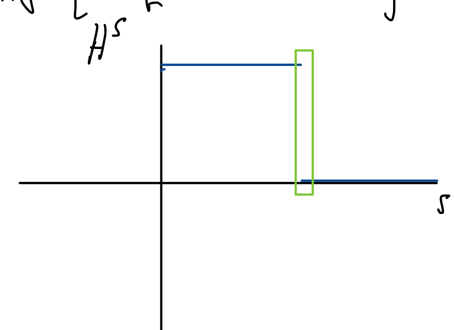
For $\delta>0$, let $H^g_\delta(A) = \inf\{\sum_{i=0}^{\infty} g(\mathrm{diam}(U_i))\colon (U_i)$ is an open cover of A by sets of diameter $<\delta\}$.

and define the g-measure H^g as $H^g(A) = \lim_{\delta\to 0} H^g_\delta(A)$.

The s-dimensional Hausdorff measure $H^s = H^g$ where $g(x)=x^s$.

The Hausdorff dimension of $A\subseteq X$ is

$$\dim_H(A) = \sup\{s\in[0,\infty) : H^s(A)=\infty\} = \inf\{s\in[0,\infty) : H^s(A)=0\}.$$



Examples and Facts:

For $\mathbb{R}^n$, H^n is usual Lebesgue measure up to a constant.

Lines in $\mathbb{R}^n$ have dimension 1.

The Cantor middle thirds set has dimension $\log 2/\log 3$.

If we let $s \to 1$, then the s-dimensional Hausdorff measures "approach" Lesbegue measure.

Definition 2.1. Suppose that f and g are gauge functions. We write $f \prec g$ if $\lim_{t\to 0^+} g(d)/f(d) = 0$ (or equivalently $\lim_{t\to 0^+} f(d)/g(d) = \infty$) and say that g has **higher order** than f.

If we take $g \prec id$, then maybe g can detect more non-reducibility than Lebesgue measure.

Marks, R., Slaman

Theorem 1.4. *Suppose $g\colon [0,\infty) \to [0,\infty)$ is a gauge function of lower order than the identity and that E is a countable Borel equivalence relation on 2^ω. Then there is a closed set $A \subseteq 2^\omega$ such that $E \restriction A$ is smooth, and $H^g(A) > 0$. In particular, there is a closed set of Hausdorff dimension 1 such that $E \restriction A$ is smooth.*

Corollary 4.3. *Suppose $g\colon [0,\infty) \to [0,\infty)$ is a gauge function of lower order than $t \mapsto t^n$ and that E is a countable Borel equivalence relation on $\mathbb{R}^n$. Then there is a closed set $A \subseteq [0,1]$ such that $E \restriction A$ is smooth, and $H^g(A) > 0$. In particular, there is a closed set of Hausdorff dimension n such that $E \restriction A$ is smooth.*

This condition is necessary:
$\equiv_T \not\leq_B E_0$ on any λ-positive set. (Slaman-Steel '88)

Marks, R., Slaman

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Other results in the same vein

Kechris, Miller

Theorem 1.1 (Hjorth-Kechris, Sullivan-Weiss-Wright, Woodin (see [KM, Theorem 12.1])). *If E is a countable Borel equivalence relation on a Polish space X, then there is a comeager invariant Borel set $C \subseteq X$ so that $E \restriction C$ is hyperfinite.*

Kanovei, Sabok, Zapletal

Theorem 1.2 (Mathias and Soare [M, So] (see [KSZ, Theorem 8.17])). *If E is a countable Borel equivalence relation on $[\omega]^\omega$, there is an $A \in [\omega]^\omega$ so that $E \restriction [A]^\omega$ is hyperfinite.*

Panagiotopoulos, Wang '22

Theorem 1.3 ([PW, Theorem 1.2]). *If E is a countable Borel equivalence relation on $(\omega)^\omega$, then there is an $A \in (\omega)^\omega$ so that $E \restriction (A)^\omega$ is smooth.*