# Cofinitary groups and projective well-orders

Lukas Schembecker

University of Hamburg
Joint work with Vera Fischer and David Schrittesser

Colloquium Logicum 2024 in Vienna

07.10.2024

# What is a (maximal) cofinitary group?

# (Maximal) cofinitary groups

**Definition**

A permutation on $\omega$ is called **finitary** if it only moves finitely many points.

# (Maximal) cofinitary groups

**Definition**

A permutation on $\omega$ is called **finitary** if it only moves finitely many points. Dually, a permutation is called **cofinitary** if only finitely many points are not moved.

# (Maximal) cofinitary groups

## Definition

A permutation on $\omega$ is called **finitary** if it only moves finitely many points. Dually, a permutation is called **cofinitary** if only finitely many points are not moved.

## Observation

The set of all finitary permutations is a group (the unique maximal group).

# (Maximal) cofinitary groups

## Definition

A permutation on $\omega$ is called **finitary** if it only moves finitely many points. Dually, a permutation is called **cofinitary** if only finitely many points are not moved.

## Observation

The set of all finitary permutations is a group (the unique maximal group). However, the set of all cofinitary permutations with the identity is not a group and indeed there are many maximal cofinitary groups.

# Sizes of maximal cofinitary groups

**Definition**

A subgroup $G \subseteq S_\omega$ is called **cofinitary** if every $g \in G \setminus \{\mathrm{id}\}$ is cofinitary. It is called **maximal** if it is maximal with respect to inclusion.

# Sizes of maximal cofinitary groups

## Definition

A subgroup $G \subseteq S_\omega$ is called **cofinitary** if every $g \in G \setminus \{\mathrm{id}\}$ is cofinitary.
It is called **maximal** if it is maximal with respect to inclusion.

## Theorem (Truss, 1985 [10]; Adeleke, 1988 [1])

*Every countable cofinitary group is not maximal.*

# Sizes of maximal cofinitary groups

**Definition**

A subgroup $G \subseteq S_\omega$ is called **cofinitary** if every $g \in G \setminus \{\mathrm{id}\}$ is cofinitary. It is called **maximal** if it is maximal with respect to inclusion.

**Theorem (Truss, 1985 [10]; Adeleke, 1988 [1])**

*Every countable cofinitary group is not maximal.*

**Theorem (Neumann)**

*There are cofinitary groups of size* $2^{\aleph_0}$.

Thus, from the perspective of combinatorial set theory we may study the possible sizes of maximal cofinitary groups:

**Definition**

$\mathfrak{a}_{\mathrm{g}} :=$ minimal size of a maximal cofinitary group.

Thus, from the perspective of combinatorial set theory we may study the possible sizes of maximal cofinitary groups:

**Definition**

$$\mathfrak{a}_{\mathrm{g}} := \text{minimal size of a maximal cofinitary group.}$$

We have just seen that $\aleph_0 < \mathfrak{a}_{\mathrm{g}} \leq 2^{\aleph_0}$, i.e. $\mathfrak{a}_{\mathrm{g}}$ is a cardinal characteristic in the usual sense. In fact:

**Theorem (Brendle, Spinas, Zhang, 2000 [2])**

$\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_g$.

# Tightness

# Separating $\mathfrak{a}_\mathrm{g}$ from other invariants

We want to construct models in which

$$\mathfrak{a}_\mathrm{g} = \aleph_1 < \aleph_2 = \mathfrak{x}$$

holds for other cardinal characteristics $\mathfrak{x}$.

# Separating $\mathfrak{a}_g$ from other invariants

We want to construct models in which

$$\mathfrak{a}_g = \aleph_1 < \aleph_2 = \mathfrak{x}$$

holds for other cardinal characteristics $\mathfrak{x}$.

**Idea (Tightness)**

We want to define a combinatorial strengthening of the maximality of a cofinitary group, which is preserved by a large class of forcing notions, e.g. by countable support iterations of certain proper forcings.

# Tightness for other families

The idea of such a combinatorial strengthening of maximality for this purpose has already been successfully developed for other types of combinatorial families:

1. Maximal almost disjoint (mad) families $\rightarrow \mathfrak{a}$
   (Guzmán, Hrušák, Téllez, 2020 [7])

# Tightness for other families

The idea of such a combinatorial strengthening of maximality for this purpose has already been successfully developed for other types of combinatorial families:

1. Maximal almost disjoint (mad) families $\rightarrow \mathfrak{a}$
   (Guzmán, Hrušák, Téllez, 2020 [7])
2. Maximal eventually different (med) families $\rightarrow \mathfrak{a}_{\mathrm{e}}$
   (Fischer, Switzer, 2023 [5])

# Tightness for other families

The idea of such a combinatorial strengthening of maximality for this purpose has already been successfully developed for other types of combinatorial families:

1. Maximal almost disjoint (mad) families $\rightarrow \mathfrak{a}$
   (Guzmán, Hrušák, Téllez, 2020 [7])
2. Maximal eventually different (med) families $\rightarrow \mathfrak{a}_e$
   (Fischer, Switzer, 2023 [5])
3. Maximal eventually different families of permutations $\rightarrow \mathfrak{a}_p$
   (Fischer, Switzer, 2023 [5])

# Tightness for other families

The idea of such a combinatorial strengthening of maximality for this purpose has already been successfully developed for other types of combinatorial families:

1. Maximal almost disjoint (mad) families $\rightarrow \mathfrak{a}$
   (Guzmán, Hrušák, Téllez, 2020 [7])
2. Maximal eventually different (med) families $\rightarrow \mathfrak{a}_{\mathsf{e}}$
   (Fischer, Switzer, 2023 [5])
3. Maximal eventually different families of permutations $\rightarrow \mathfrak{a}_{\mathsf{p}}$
   (Fischer, Switzer, 2023 [5])

It turns out that the last notion of tightness may also be used in the context of cofinitary groups $\rightarrow \mathfrak{a}_{\mathrm{g}}$.

# Preserving tightness

## Definition

Let $G$ be a cofinitary group. Then we say $G$ is **tight** if $G$ is tight as an eventually different family of permutations.

# Preserving tightness

## Definition

Let $G$ be a cofinitary group. Then we say $G$ is **tight** if $G$ is tight as an eventually different family of permutations.

## Theorem (Fischer, Switzer, 2023 [5])

*Assume $F$ is a tight family of permutations and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma \rangle$ is a countable support iteration of proper forcings such that for every $\alpha < \gamma$ we have*

$$\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha \text{ strongly preserves the tightness of } F.$$

*Then $\mathbb{P}_\gamma$ strongly preserves the tightness of $F$.*

# Preserving tightness

## Definition

Let $G$ be a cofinitary group. Then we say $G$ is **tight** if $G$ is tight as an eventually different family of permutations.

## Theorem (Fischer, Switzer, 2023 [5])

*Assume $F$ is a tight family of permutations and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma \rangle$ is a countable support iteration of proper forcings such that for every $\alpha < \gamma$ we have*

$$\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha \text{ strongly preserves the tightness of } F.$$

*Then $\mathbb{P}_\gamma$ strongly preserves the tightness of $F$.*

## Theorem (Fischer, Switzer, 2023 [5])

*Miller forcing, Miller partition forcing, Shelah's forcing $\mathbb{Q}_{\mathcal{I}}$, ... all strongly preserve the tightness of every tight family of permutations.*

# Separating $\mathfrak{a}_g$ from other invariants

**Theorem (Fischer, S., Schrittesser, 2023 [3])**

*Under* MA*(*$\sigma$*-centered) every cofinitary group of size* $<2^{\aleph_0}$ *can be embedded into a tight cofinitary group of size* $2^{\aleph_0}$*.*

# Separating $\mathfrak{a}_\mathrm{g}$ from other invariants

**Theorem (Fischer, S., Schrittesser, 2023 [3])**

*Under* MA*($\sigma$-centered) every cofinitary group of size* $<2^{\aleph_0}$ *can be embedded into a tight cofinitary group of size* $2^{\aleph_0}$.

**Theorem (Fischer, S., Schrittesser, 2023 [3])**

*The following constellations are consistent with a tight witness for* $\mathfrak{a}_g$*:*

1. $\mathfrak{a}_g = \mathfrak{d} = \mathfrak{a}_T < \mathfrak{c} = \aleph_2$,
2. $\mathfrak{a}_g < \mathfrak{d} = \mathfrak{a}_T = \mathfrak{c} = \aleph_2$,
3. $\mathfrak{a}_g = \mathfrak{d} < \mathfrak{a}_T = \mathfrak{c} = \aleph_2$,
4. $\mathfrak{a}_g = \mathfrak{i} < \mathfrak{u} = \mathfrak{c} = \aleph_2$,
5. ...

# Optimal projective complexity

# Optimal witnesses for $\mathfrak{a}_g$

We have seen that ZFC implies the existence of maximal cofinitary groups.
What is the optimal projective complexity for such groups?

# Optimal witnesses for $\mathfrak{a}_{\mathrm{g}}$

We have seen that ZFC implies the existence of maximal cofinitary groups. What is the optimal projective complexity for such groups?

**Observation**

In $L$ there is a maximal cofinitary group with a $\Sigma^1_2$-set of generators.

# Optimal witnesses for $\mathfrak{a}_g$

We have seen that ZFC implies the existence of maximal cofinitary groups. What is the optimal projective complexity for such groups?

**Observation**

In $L$ there is a maximal cofinitary group with a $\Sigma^1_2$-set of generators.

**Theorem (Gao, Zhang, 2008 [6])**

*In $L$ there is a maximal cofinitary group with a $\Pi^1_1$-set of generators.*

# Optimal witnesses for $\mathfrak{a}_g$

We have seen that ZFC implies the existence of maximal cofinitary groups. What is the optimal projective complexity for such groups?

**Observation**

In $L$ there is a maximal cofinitary group with a $\Sigma^1_2$-set of generators.

**Theorem (Gao, Zhang, 2008 [6])**

*In L there is a maximal cofinitary group with a $\Pi^1_1$-set of generators.*

**Theorem (Kastermans, 2009 [9])**

*In L there is a $\Pi^1_1$ maximal cofinitary group.*

# Optimal witnesses for $\mathfrak{a}_g$

**Theorem (Horowitz, Shelah, 2016 [8])**

ZF*: There is a Borel maximal cofinitary group.*

# Optimal witnesses for $\mathfrak{a}_g$

**Theorem (Horowitz, Shelah, 2016 [8])**

ZF*: There is a Borel maximal cofinitary group.*

**Theorem (Fischer, Schrittesser, Törnquist, 2017 [4])**

*In L there is a* $\Pi^1_1$ *Cohen-indestructible maximal cofinitary group.*

# Optimal witnesses for $\mathfrak{a}_g$

## Theorem (Horowitz, Shelah, 2016 [8])

ZF: *There is a Borel maximal cofinitary group.*

## Theorem (Fischer, Schrittesser, Törnquist, 2017 [4])

*In L there is a $\Pi^1_1$ Cohen-indestructible maximal cofinitary group.*

## Corollary (Fischer, Schrittesser, Törnquist, 2017 [4])

*The following constellation is consistent with a $\Pi^1_1$ witness for $\mathfrak{a}_g$:*

$$\aleph_1 = \mathfrak{a}_g = \mathfrak{b} < \mathfrak{d} = 2^{\aleph_0}$$

# Optimal witnesses for $\mathfrak{a}_{\mathrm{g}}$

**Theorem (Fischer, S., Schrittesser, 2023 [3])**

*In $L$ there is a $\Pi^1_1$ tight cofinitary group.*

# Optimal witnesses for $\mathfrak{a}_\mathrm{g}$

**Theorem (Fischer, S., Schrittesser, 2023 [3])**

*In L there is a $\Pi^1_1$ tight cofinitary group.*

**Theorem (Fischer, S., Schrittesser, 2023 [3])**

*The following constellations are consistent with a $\Pi^1_1$ tight witness for $\mathfrak{a}_g$:*

1. $\mathfrak{a}_g = \mathfrak{d} = \mathfrak{a}_T < \mathfrak{c} = \aleph_2$,
2. $\mathfrak{a}_g < \mathfrak{d} = \mathfrak{a}_T = \mathfrak{c} = \aleph_2$,
3. $\mathfrak{a}_g = \mathfrak{d} < \mathfrak{a}_T = \mathfrak{c} = \aleph_2$,
4. $\mathfrak{a}_g = \mathfrak{i} < \mathfrak{u} = \mathfrak{c} = \aleph_2$.

# Optimal witnesses for $\mathfrak{a}_g$

**Theorem (Fischer, S., Schrittesser, 2023 [3])**

*In L there is a $\Pi^1_1$ tight cofinitary group.*

**Theorem (Fischer, S., Schrittesser, 2023 [3])**

*The following constellations are consistent with a $\Pi^1_1$ tight witness for $\mathfrak{a}_g$:*

1. $\mathfrak{a}_g = \mathfrak{d} = \mathfrak{a}_T < \mathfrak{c} = \aleph_2$,
2. $\mathfrak{a}_g < \mathfrak{d} = \mathfrak{a}_T = \mathfrak{c} = \aleph_2$,
3. $\mathfrak{a}_g = \mathfrak{d} < \mathfrak{a}_T = \mathfrak{c} = \aleph_2$,
4. $\mathfrak{a}_g = \mathfrak{i} < \mathfrak{u} = \mathfrak{c} = \aleph_2$.

**Corollary (Fischer, S., Schrittesser, 2023 [3])**

*All models above may also contain a $\Delta^1_3$-definable well-order of the reals.*

# Coding into orbits

# Constructing a cofinitary group with a $\Pi^1_1$-set of generators

The construction in L of a tight cofinitary group with a $\Pi^1_1$-set of generators boils down to the following inductive construction of $\langle G_\alpha \mid \alpha < \aleph_1 \rangle$:

# Constructing a cofinitary group with a $\Pi^1_1$-set of generators

The construction in L of a tight cofinitary group with a $\Pi^1_1$-set of generators boils down to the following inductive construction of $\langle G_\alpha \mid \alpha < \aleph_1 \rangle$:

1. Given a countable cofinitary group $G_\alpha \in L_{\delta_\alpha}$ and a real number $r$ we need to find a cofinitary $f \in S_\infty \setminus G_\alpha$, such that $G_{\alpha+1} := \langle G_\alpha \cup \{f\} \rangle$ is a cofinitary group and $r$ can be decoded from $f$.

# Constructing a cofinitary group with a $\Pi^1_1$-set of generators

The construction in L of a tight cofinitary group with a $\Pi^1_1$-set of generators boils down to the following inductive construction of $\langle G_\alpha \mid \alpha < \aleph_1 \rangle$:

1. Given a countable cofinitary group $G_\alpha \in L_{\delta_\alpha}$ and a real number $r$ we need to find a cofinitary $f \in S_\infty \setminus G_\alpha$, such that $G_{\alpha+1} := \langle G_\alpha \cup \{f\} \rangle$ is a cofinitary group and $r$ can be decoded from $f$.
2. In order to find such an $f$ we will use 'forcing' over the countable model $L_\delta$, i.e. $f$ will be a generic for Zhang's forcing over $L_\delta$.

# Constructing a cofinitary group with a $\Pi^1_1$-set of generators

The construction in L of a tight cofinitary group with a $\Pi^1_1$-set of generators boils down to the following inductive construction of $\langle G_\alpha \mid \alpha < \aleph_1 \rangle$:

1. Given a countable cofinitary group $G_\alpha \in L_{\delta_\alpha}$ and a real number $r$ we need to find a cofinitary $f \in S_\infty \setminus G_\alpha$, such that $G_{\alpha+1} := \langle G_\alpha \cup \{f\} \rangle$ is a cofinitary group and $r$ can be decoded from $f$.
2. In order to find such an $f$ we will use 'forcing' over the countable model $L_\delta$, i.e. $f$ will be a generic for Zhang's forcing over $L_\delta$.
3. Using the $\Delta^1_2$ well-order in L we can make sure that $f$ is picked uniquely in some countable $L_{\delta_{\alpha+1}}$.

# Constructing a cofinitary group with a $\Pi^1_1$-set of generators

The construction in L of a tight cofinitary group with a $\Pi^1_1$-set of generators boils down to the following inductive construction of $\langle G_\alpha \mid \alpha < \aleph_1 \rangle$:

1. Given a countable cofinitary group $G_\alpha \in L_{\delta_\alpha}$ and a real number $r$ we need to find a cofinitary $f \in S_\infty \setminus G_\alpha$, such that $G_{\alpha+1} := \langle G_\alpha \cup \{f\} \rangle$ is a cofinitary group and $r$ can be decoded from $f$.
2. In order to find such an $f$ we will use 'forcing' over the countable model $L_\delta$, i.e. $f$ will be a generic for Zhang's forcing over $L_\delta$.
3. Using the $\Delta^1_2$ well-order in L we can make sure that $f$ is picked uniquely in some countable $L_{\delta_{\alpha+1}}$.
4. Finally $G := \bigcup_{\alpha < \aleph_1}$ should be a tight cofinitary group.

# Constructing a cofinitary group with a $\Pi^1_1$-set of generators

In summary we have adapt Zhang's forcing, so that its generic real $f_{\text{gen}}$ codes a real $r$ and the iteration yields a tight cofinitary group.

# Constructing a cofinitary group with a $\Pi^1_1$-set of generators

In summary we have adapt Zhang's forcing, so that its generic real $f_{\text{gen}}$ codes a real $r$ and the iteration yields a tight cofinitary group.

**Theorem (Fischer, S., Schrittesser, 2023 [3])**

*Iterating Zhang's forcing yields a tight cofinitary group.*

# Constructing a cofinitary group with a $\Pi^1_1$-set of generators

In summary we have adapt Zhang's forcing, so that its generic real $f_{\text{gen}}$ codes a real $r$ and the iteration yields a tight cofinitary group.

**Theorem (Fischer, S., Schrittesser, 2023 [3])**

*Iterating Zhang's forcing yields a tight cofinitary group.*

Previous coding techniques directly coded $r$ into the function values of the generic real $f_{\text{gen}}$. However, in order to obtain a tight cofinitary group we had to come up with a more flexible coding technique.

# Orbits of permutations

Given a permutation $f \in S_\omega$ we may consider the lengths of its orbits:

## Definition

An orbit of $f$ is a minimal non-empty subset of $\omega$ closed under the applications of $f$ and $f^{-1}$.

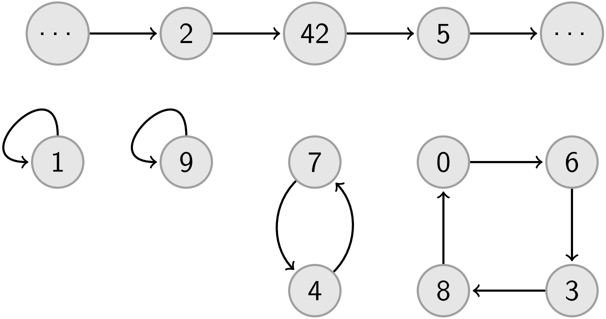
# Orbits of permutations

Given a permutation $f \in S_\omega$ we may consider the lengths of its orbits:

**Definition**

An orbit of $f$ is a minimal non-empty subset of $\omega$ closed under the applications of $f$ and $f^{-1}$.

Hence, any permutation $f \in S_\omega$ partitions $\omega$ into its finite and infinite orbits:



# Orbits of the Zhang-generic

**Proposition (Fischer, S., Schrittesser, 2023 [3])**

*The Zhang generic $f_{gen}$ only has finite orbits.*

# Orbits of the Zhang-generic

**Proposition (Fischer, S., Schrittesser, 2023 [3])**

*The Zhang generic $f_{gen}$ only has finite orbits.*

**Definition**

Given $f \in \omega^\omega$ let $\mathcal{O}_f$ be the set of all orbits of $f$.

# Orbits of the Zhang-generic

## Proposition (Fischer, S., Schrittesser, 2023 [3])

*The Zhang generic $f_{gen}$ only has finite orbits.*

## Definition

Given $f \in \omega^\omega$ let $\mathcal{O}_f$ be the set of all orbits of $f$. There is a natural well-order on $\mathcal{O}_f$ defined for $O, P \in \mathcal{O}_f$ by $O < P$ iff $\min(O) < \min(P)$.

# Orbits of the Zhang-generic

## Proposition (Fischer, S., Schrittesser, 2023 [3])

*The Zhang generic $f_{gen}$ only has finite orbits.*

## Definition

Given $f \in \omega^\omega$ let $\mathcal{O}_f$ be the set of all orbits of $f$. There is a natural well-order on $\mathcal{O}_f$ defined for $O, P \in \mathcal{O}_f$ by $O < P$ iff $\min(O) < \min(P)$. Assume $f$ only has finite orbits; it follows that $f$ has infinitely many orbits.

# Orbits of the Zhang-generic

**Proposition (Fischer, S., Schrittesser, 2023 [3])**

*The Zhang generic $f_{gen}$ only has finite orbits.*

**Definition**

Given $f \in \omega^\omega$ let $\mathcal{O}_f$ be the set of all orbits of $f$. There is a natural well-order on $\mathcal{O}_f$ defined for $O, P \in \mathcal{O}_f$ by $O < P$ iff $\min(O) < \min(P)$. Assume $f$ only has finite orbits; it follows that $f$ has infinitely many orbits. Then, we define its **orbit coding function** $o_f : \omega \to 2$ by

$$o_f(n) := (|O_n| \mod 2),$$

where $O_n$ is the $n$-th element in the well-order of $\mathcal{O}_f$.

# A $\Pi^1_1$ tight cofinitary group

**Theorem (Fischer, S., Schrittesser, 2023 [3])**

*Zhang's forcing may be adapted so that $r = o_{f_{gen}}$ and iterating the forcing yields a cofinitary group.*

# A $\Pi^1_1$ tight cofinitary group

**Theorem (Fischer, S., Schrittesser, 2023 [3])**

*Zhang's forcing may be adapted so that $r = o_{f_{gen}}$ and iterating the forcing yields a cofinitary group.*

**Corollary (Fischer, S., Schrittesser, 2023 [3])**

*In $L$ there is a tight cofinitary group with a $\Pi^1_1$-set of generators.*

# A $\Pi^1_1$ tight cofinitary group

**Theorem (Fischer, S., Schrittesser, 2023 [3])**

*Zhang's forcing may be adapted so that $r = o_{f_{gen}}$ and iterating the forcing yields a cofinitary group.*

**Corollary (Fischer, S., Schrittesser, 2023 [3])**

*In $L$ there is a tight cofinitary group with a $\Pi^1_1$-set of generators.*

With the coding above we do not obtain that the full group is $\Pi^1_1$.

# A $\Pi^1_1$ tight cofinitary group

**Theorem (Fischer, S., Schrittesser, 2023 [3])**

*Zhang's forcing may be adapted so that $r = o_{f_{gen}}$ and iterating the forcing yields a cofinitary group.*

**Corollary (Fischer, S., Schrittesser, 2023 [3])**

*In $L$ there is a tight cofinitary group with a $\Pi^1_1$-set of generators.*

With the coding above we do not obtain that the full group is $\Pi^1_1$. However, one may adapt the orbit coding function, so that every new real in the group $\langle G \cup \{f_{\text{gen}}\}\rangle$ codes $r$.

**Proposition (Fischer, S., Schrittesser, 2023 [3])**

*For every $k$ the Zhang generic $f_{gen}$ only has finitely many orbits of length $k$.*

# A $\Pi^1_1$ tight cofinitary group

**Theorem (Fischer, S., Schrittesser, 2023 [3])**

*Zhang's forcing may be adapted so that $r = o_{f_{gen}}$ and iterating the forcing yields a cofinitary group.*

**Corollary (Fischer, S., Schrittesser, 2023 [3])**

*In $L$ there is a $\Pi^1_1$ tight cofinitary group.*

With the coding above we do not obtain that the full group is $\Pi^1_1$. However, one may adapt the orbit coding function, so that every new real in the group $\langle G \cup \{f_{\text{gen}}\}\rangle$ codes $r$.

**Proposition (Fischer, S., Schrittesser, 2023 [3])**

*For every $k$ the Zhang generic $f_{gen}$ only has finitely many orbits of length $k$.*

# References I


S. A. Adeleke.
Embeddings of infinite permutation groups in sharp, highly transitive, and homogeneous groups.
*Proceedings of the Edinburgh Mathematical Society*, 31(2):169–178, 1988.

Jörg Brendle, Otmar Spinas, and Yi Zhang.
Uniformity of the Meager Ideal and Maximal Cofinitary Groups.
*Journal of Algebra*, 232:209–225, 2000.

Vera Fischer, Lukas Schembecker, and David Schrittesser.
Cofinitary groups and projective well-orders.
preprint, 2023.

Vera Fischer, David Schrittesser, and Asger Törnquist.
A co-analytic Cohen indestructible maximal cofinitary group.
*Journal of Symbolic Logic*, 82(2):629–647, 2017.

Vera Fischer and Corey Switzer.
Tight eventually different families.
*Journal of Symbolic Logic*, 2023.

Su Gao and Yi Zhang.
Definable sets of generators in maximal cofinitary groups.
*Advances in Mathematics*, 217(2):814–832, 2008.

Osvaldo Guzmán, Michael Hrušák, and Osvaldo Téllez.
Restricted MAD families.
*Journal of Symbolic Logic*, 85:149–165, 2020.

# References II


Haim Horowitz and Saharon Shelah.
A Borel maximal cofinitary group.
*The Journal of Symbolic Logic*, pages 1–14, 2023.

Bart Kastermans.
The Complexity of Maximal Cofinitary Groups.
*Proceedings of the American Mathematical Society*, 137(1):307–316, 2009.

J. K. Truss.
Embeddings of infinite permutation groups.
*Proceedings of Groups - St Andrews*, pages 335–351, 1985.

Thank you for your attention!