TECHNISCHE UNIVERSITÄT DRESDEN

Digraphs modulo primitive positive constructability

Florian Starke

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European Research Council
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Introduction – Constraint Satisfaction Problems

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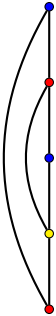

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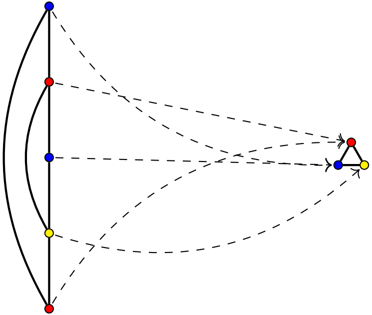

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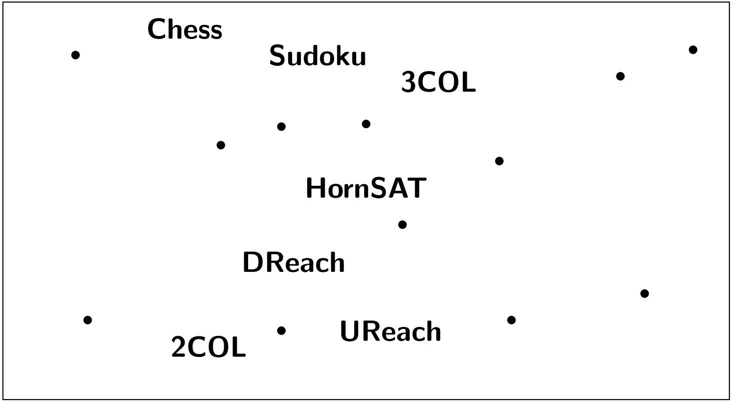

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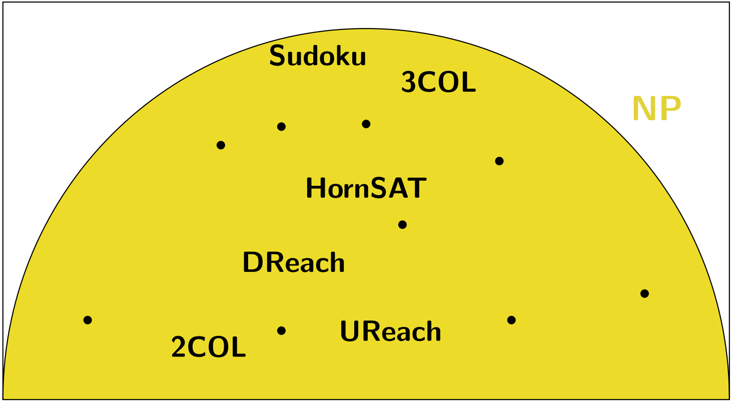

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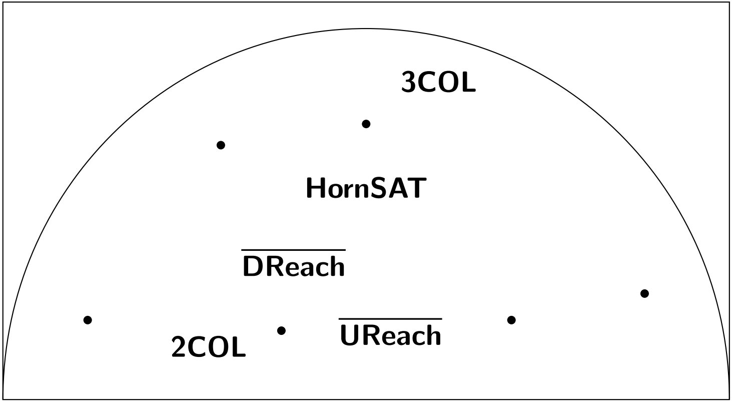

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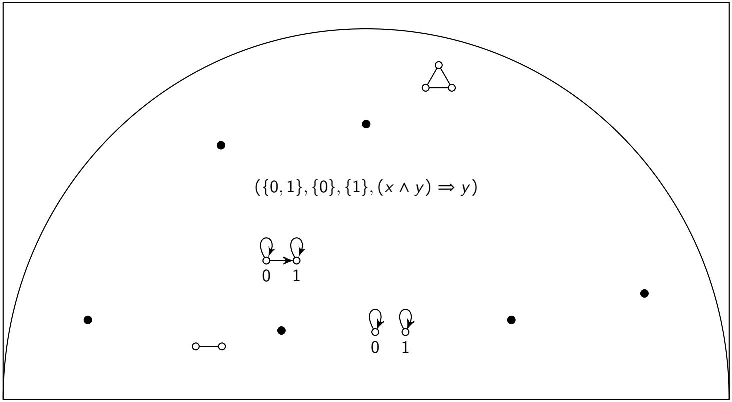

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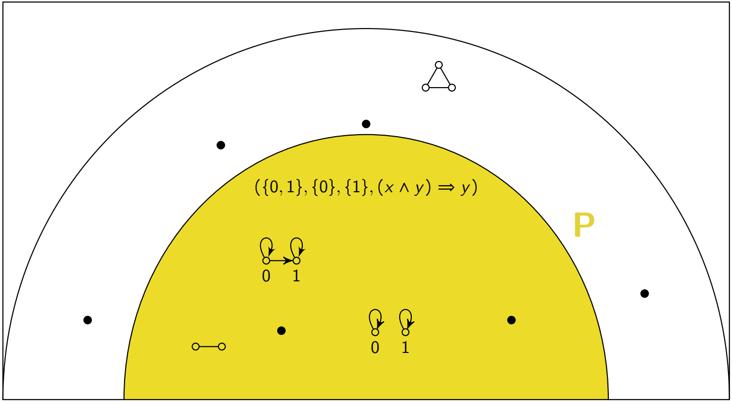

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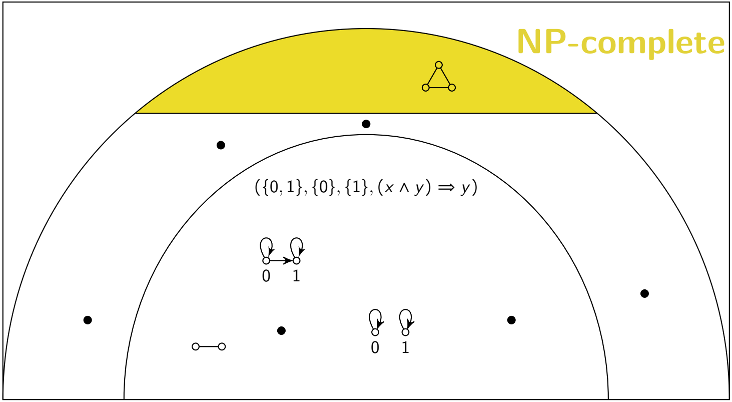

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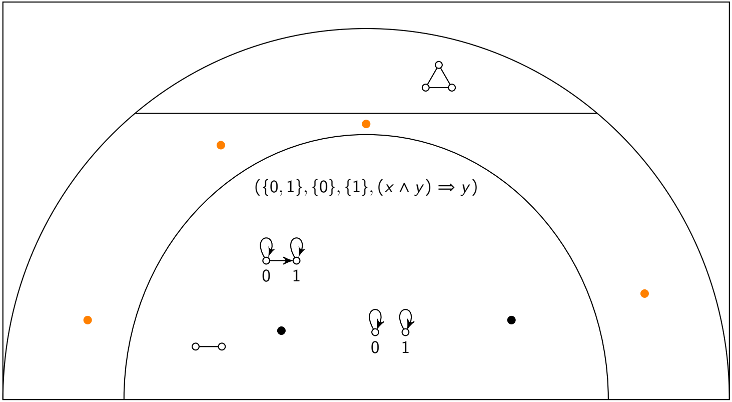

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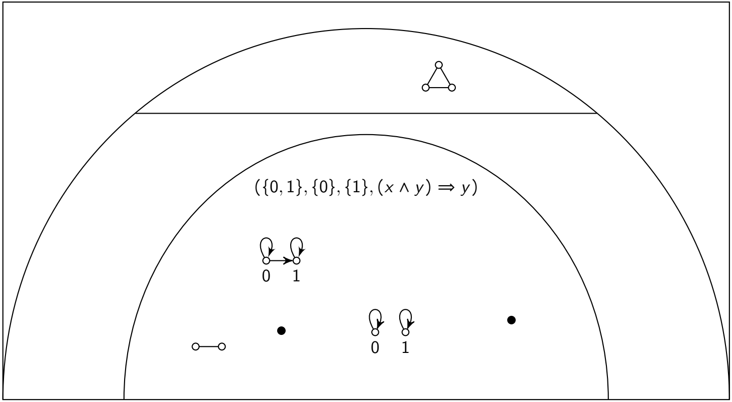
Theorem (Bulatov, Zhuk 2017): P-NP intermediate CSPs do not exist.



Introduction – PP-Construction Poset

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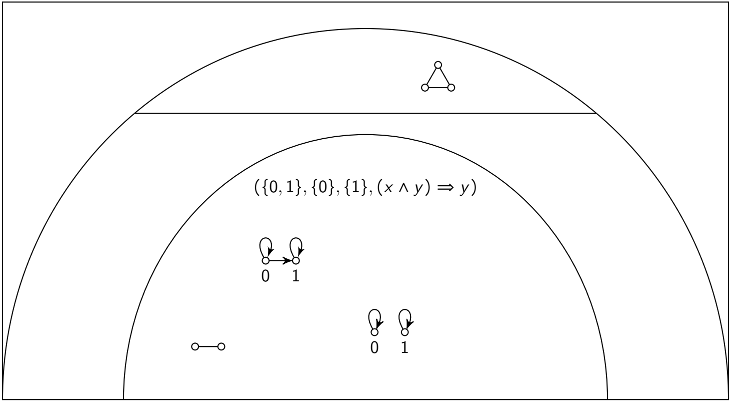

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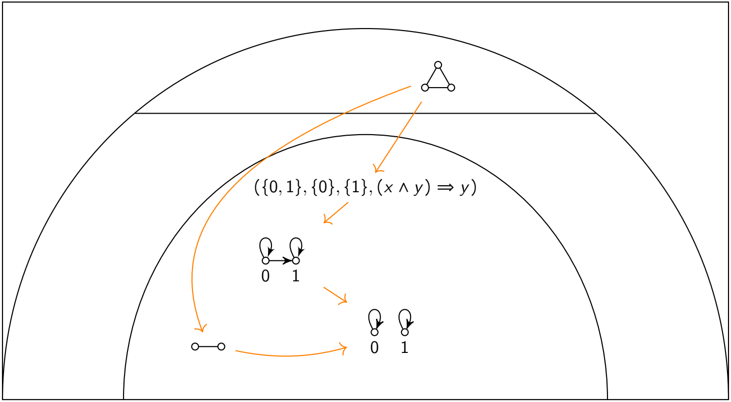

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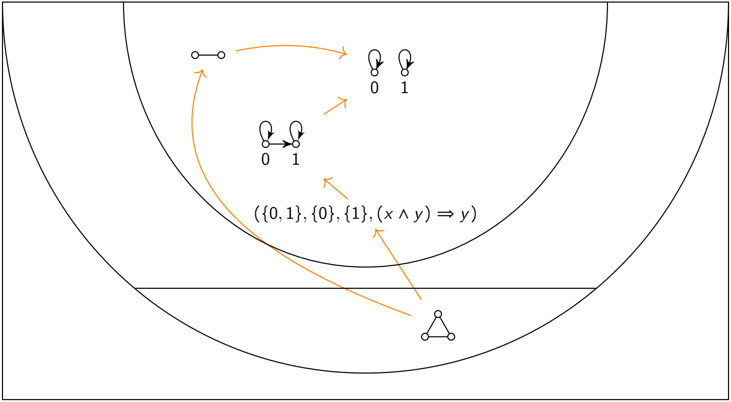

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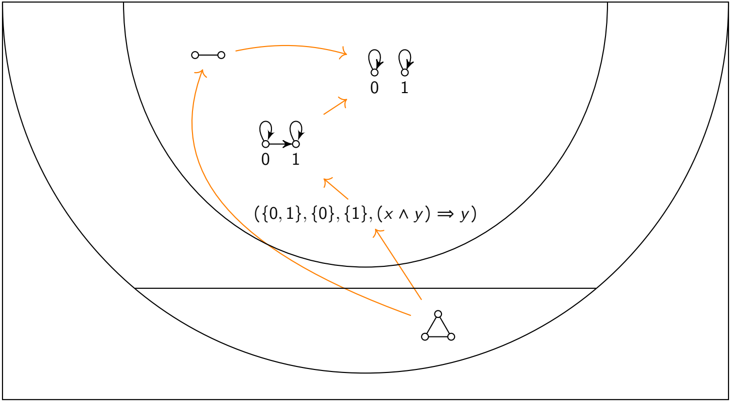
Goal: Understand the complexity of CSPs within P.



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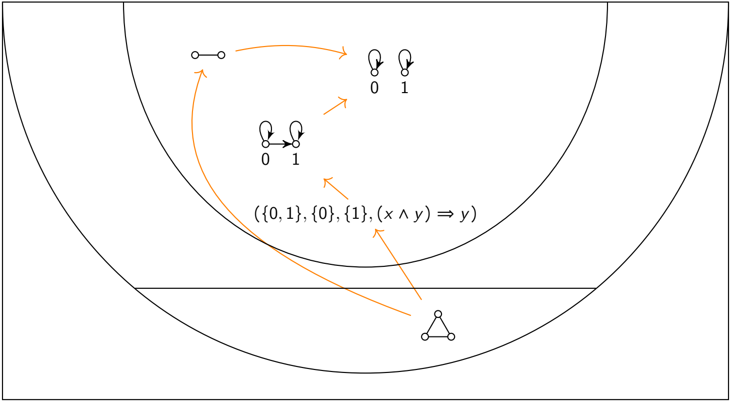
Goal: Understand the poset of finite structures ordered by $\leq_{pp}$.



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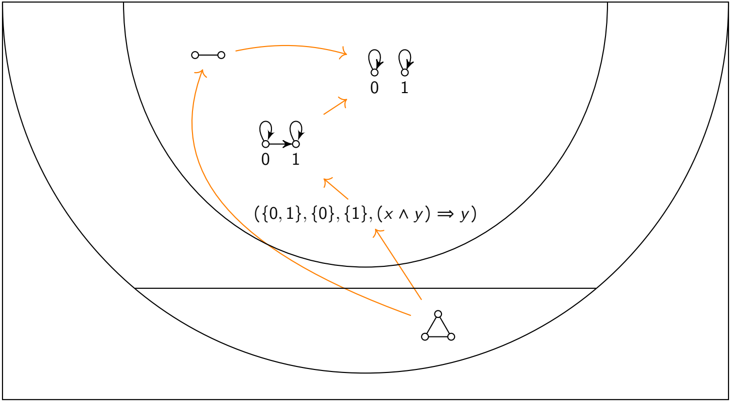
Theorem (Bulín, Delic, Jackson, Niven 2015): For all structures $\mathbb{A}$ there exists a digraph $\mathbb{G}$: $\mathrm{CSP}(\mathbb{A}) \equiv_{\text{log-space}} \mathrm{CSP}(\mathbb{G})$.



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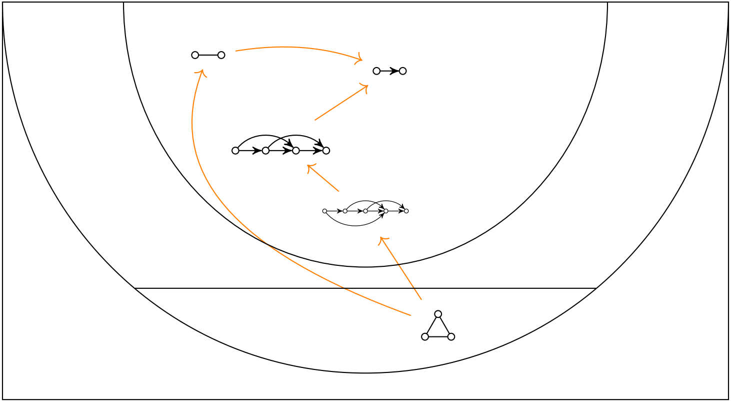

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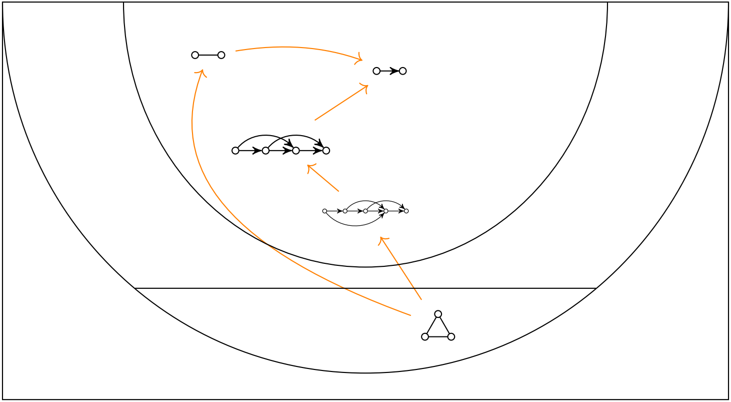

Overview – Thesis

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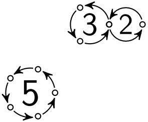
Smooth Digraphs



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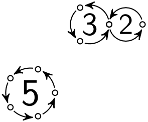
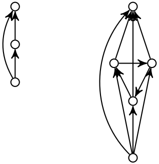


Tournaments



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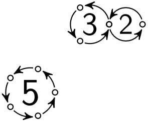

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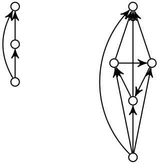
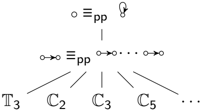


Digraphs at the top of $\mathfrak{P}_{\text{Digraphs}}$



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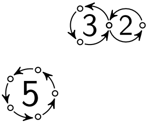

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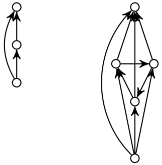

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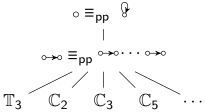
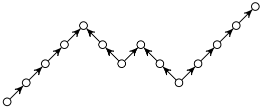


Orientations of Paths



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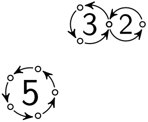

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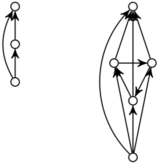

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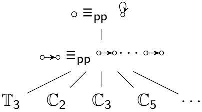

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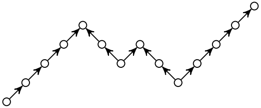
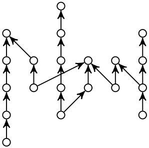


Orientations of Trees



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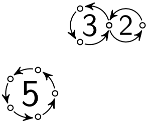

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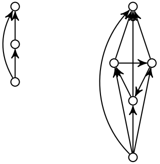

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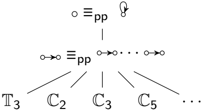

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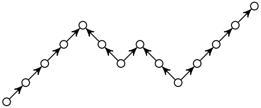

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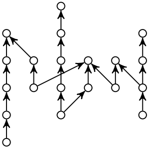
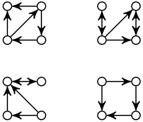


Digraphs with at most 4 vertices



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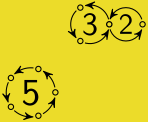

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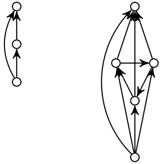

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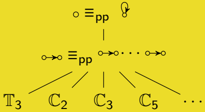

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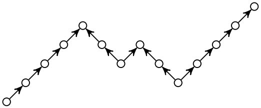

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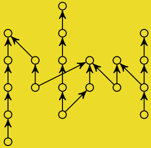

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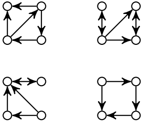

PP-Constructions – Example 1

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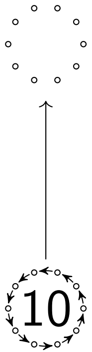
$$\Phi_E(x, y) = \exists z.\ x \rightarrow z$$
$$\wedge z \rightarrow y$$



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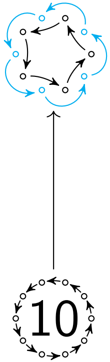

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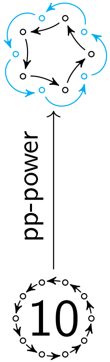

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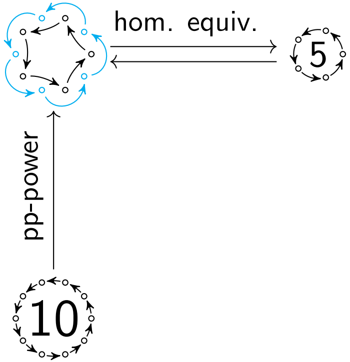

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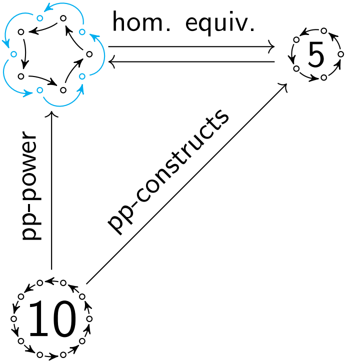

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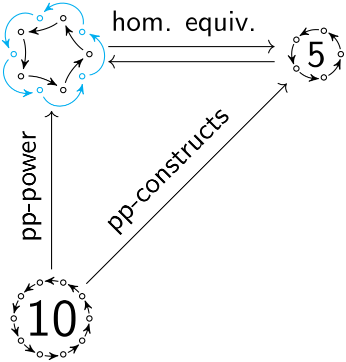

$$\mathbb{C}_{10} \leq_{\text{pp}} \mathbb{C}_5$$

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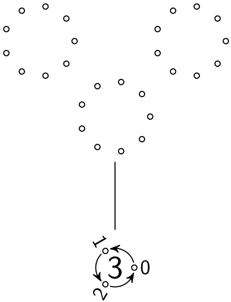

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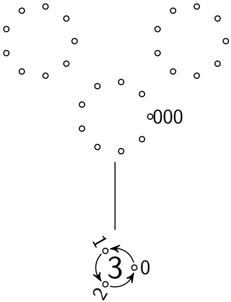

PP-Constructions – Example 2

$$\Phi_E\begin{pmatrix} x_1, x_2, x_3, \\ y_1, y_2, y_3 \end{pmatrix} = x_1 \rightarrow y_3$$
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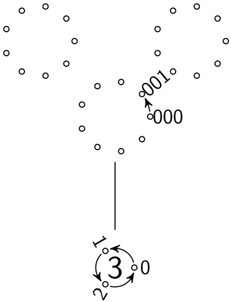

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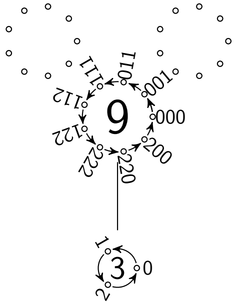

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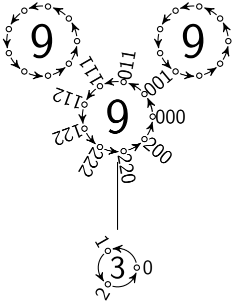

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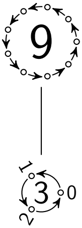

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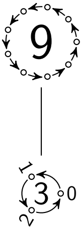

$$\mathbb{C}_3 \leq_{\mathrm{pp}} \mathbb{C}_9$$

PP-Constructions – Poset

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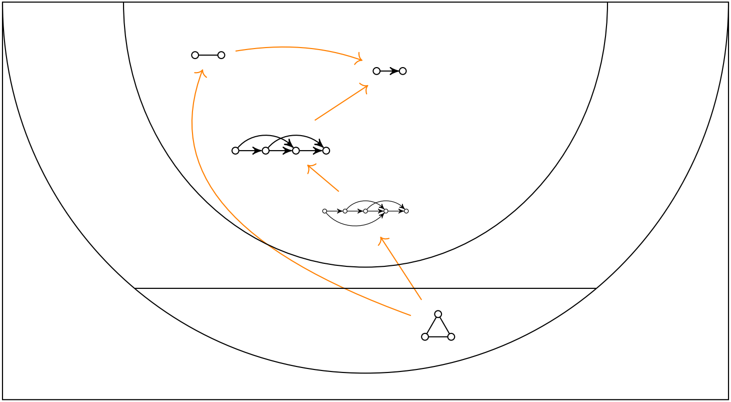

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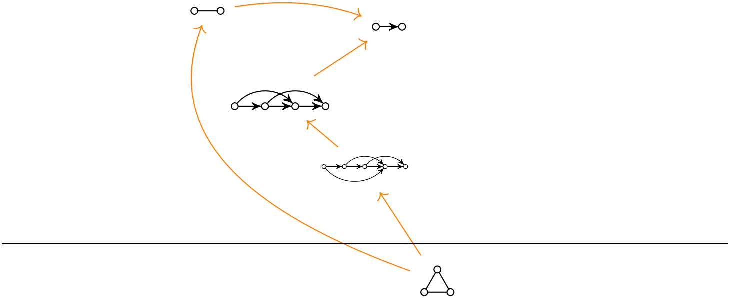

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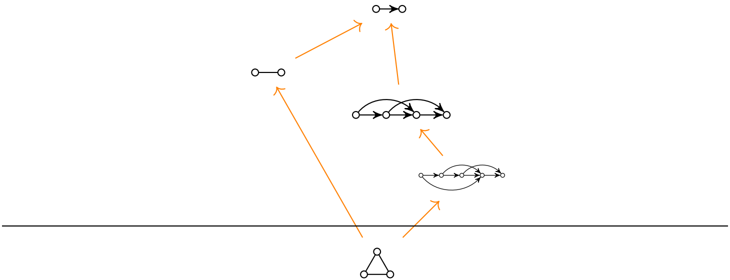

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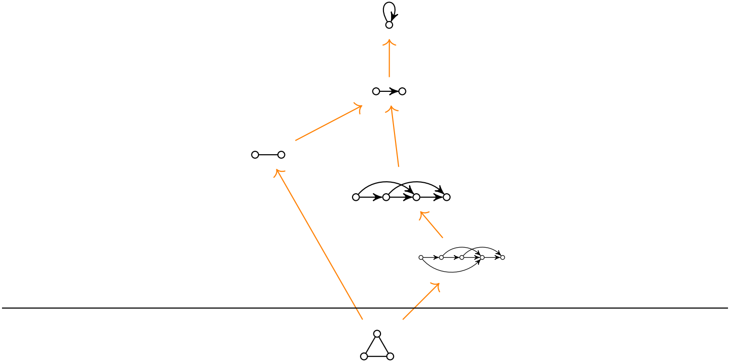

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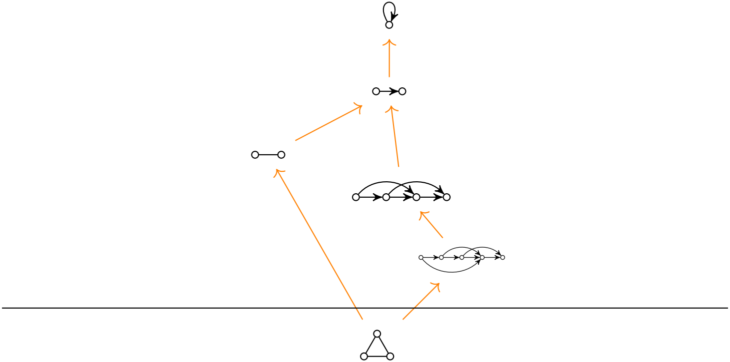
For all structures $\mathbb{A}$:  $\leq_{pp}$ $\mathbb{A}$ $\leq_{pp}$ 



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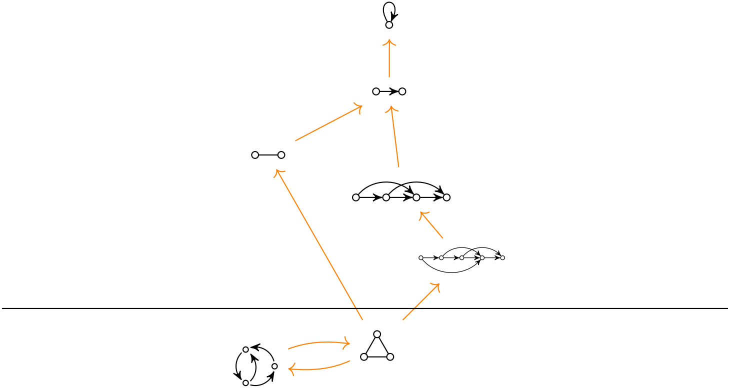

Smooth Digraphs

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Theorem (Barto, Kozik, Niven 2009): Let $\mathbb{G}$ be a smooth digraph. Then exactly one of the following is true:

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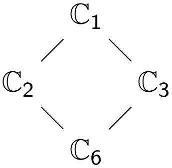

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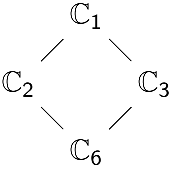

$\mathbb{C}_{2,3}$ $\quad$ $\mathbb{C}_{4,6}$ $\quad$ $\mathbb{C}_{6,8,9}$

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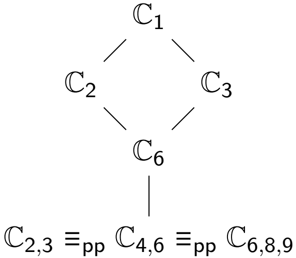

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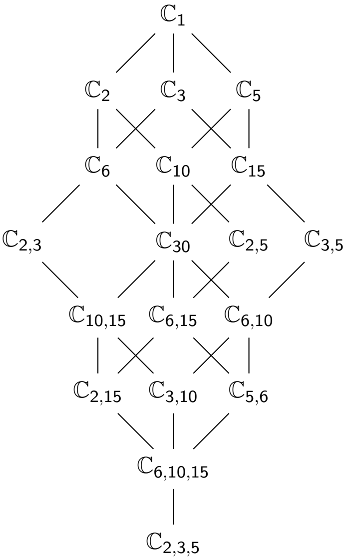

Exploring the Upper Levels of $\mathfrak{P}_{\text{Digraphs}}$

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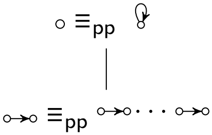

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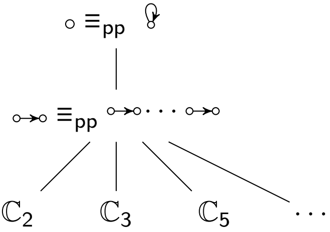

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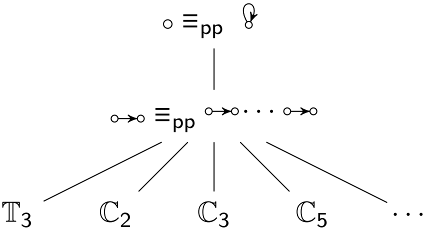
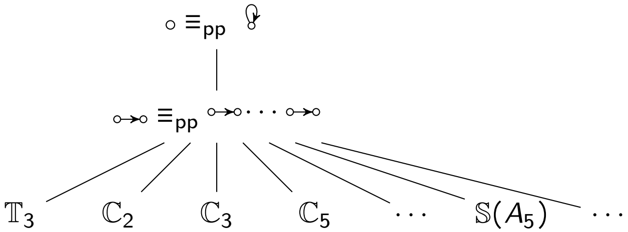
Theorem (Bodirsky, Starke 2021): The lower covers of $\circ\!\rightarrow\!\circ$ are $\mathbb{T}_3, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_5, \ldots$.



Exploring the Upper Levels of $\mathfrak{P}_{\text{Digraphs}}$

Theorem (Meyer, Starke 2024): The lower covers of ∘→∘ in the poset of all finite structures are $\mathbb{T}_3, \mathbb{S}(G_1), \mathbb{S}(G_2), \ldots$, where $G_1, G_2, \ldots$ are all finite simple groups.



Smallest Hard Trees

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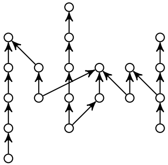

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Gutjahr	1991	81
Hell, Nešetřil, and Zhu	1996	45
Barto, Kozik, Maróti, and Niven	2009	39
Fischer	2015	30
Tatarko	2019	26

Smallest Hard Trees

n	trees	core trees	total time
10	24635	36	13 ms
11	108968	85	33 ms
12	492180	226	84 ms
13	2266502	578	236 ms
14	10598452	1569	657 ms
15	50235931	4243	2.0 s
16	240872654	11848	5.7 s
17	1166732814	33104	16.6 s
18	5702001435	94221	49.3 s
19	28088787314	269455	2.5 min
20	139354922608	779268	7.4 min

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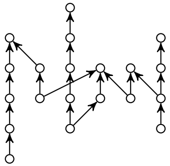

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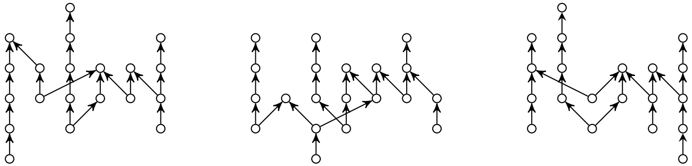

Conclusion

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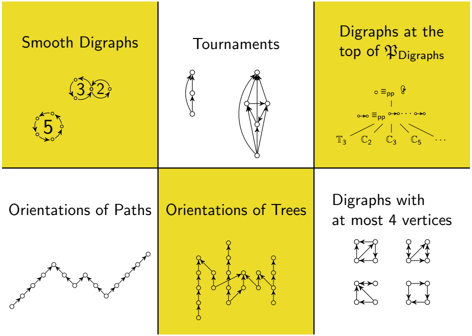

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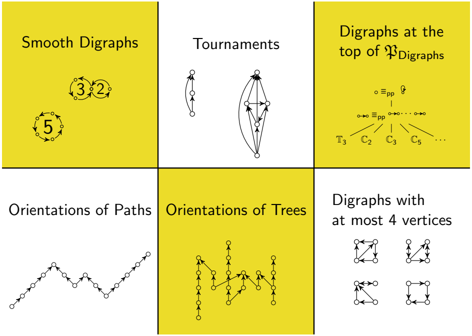

Some Open Problems

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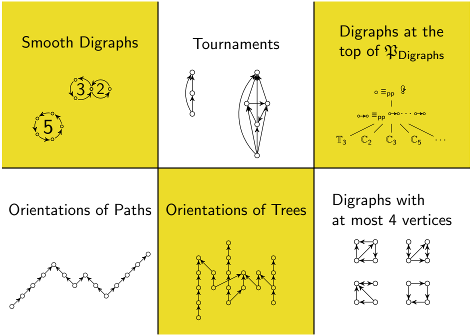

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Thank You!