The Architecture of Mathematics Revisited

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Nicholas Bourbaki (?)

The Article

THE ARCHITECTURE OF MATHEMATICS*

NICHOLAS BOURBAKI†

1. Mathematic or mathematics? To present a view of the entire field of mathematical science as it exists,—this is an enterprise which presents, at first sight, almost insurmountable difficulties, on account of the extent and the varied character of the subject. As is the case in all other sciences, the number of mathematicians and the number of works devoted to mathematics have greatly increased since the end of the 19th century. The memoirs in pure mathematics published in the world during a normal year cover several thousands of pages. Of course, not all of this material is of equal value; but, after full allowance has been made for the unavoidable tares, it remains true nevertheless that mathematical science is enriched each year by a mass of new results, that it spreads and branches out steadily into theories, which are subjected to modifications based on new foundations, compared and combined with one another. No mathematician, even were he to devote all his time to the task, would be able to follow all the details of this development. Many mathematicians take up quarters in a corner of the domain of mathematics, which they do not intend to leave; not only do they ignore almost completely what does not concern their special field, but they are unable to understand the language and the terminology used by colleagues who are working in a corner remote from their own. Even among those who have the widest training, there are none who do not feel lost in certain regions of the immense world of mathematics; those who, like Poincaré or Hilbert, put the seal of their genius on almost every domain, constitute a very great exception even among the men of greatest accomplishment.

It must therefore be out of the question to give to the uninitiated an exact picture of that which the mathematicians themselves can not conceive in its totality. Nevertheless it is legitimate to ask whether this exuberant proliferation makes for the development of a strongly constructed organism, acquiring ever greater cohesion and unity with its new growths, or whether it is the external manifestation of a tendency towards a progressive splintering, inherent in the very nature of mathematics, whether the domain of mathematics is not becoming a tower of Babel, in which autonomous disciplines are being more and more widely separated from one another, not only in their aims, but also in their methods and even in their language. In other words, do we have today a mathematic or do we have several mathematics?

Although this question is perhaps of greater urgency now than ever before, it is by no means a new one; it has been asked almost from the very beginning of mathematical science. Indeed, quite apart from applied mathematics, there has

* Authorized translation by Arnold Dresden of a chapter in "Les grands courants de la pensée mathématique," edited by F. Le Lionnais (Cahiers du Sud, 1948).

† "Professor N. Bourbaki, formerly of the Royal Poldavian Academy, now residing in Nancy, France, is the author of a comprehensive treatise of modern mathematics, in course of publication under the title *Eléments de Mathématique* (Hermann et Cie, Paris 1939–), of which ten volumes have appeared so far."

Bourbaki's Structuralism (BS)

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[T]o define a **structure***, one takes as given one or several relations, into which these elements enter [...]; then one postulates that the given relation, or relations, satisfy certain conditions (which are explicitly stated and which are the* **axioms of the structure** *under consideration) (ibid., p. 225-6)*

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At the center of our universe are found the greatest types of structures; they might be called the **mother-structures***. [...] Beyond the first nucleus, appear the structures which might be called* **multiple structures***. [...] Farther along we come to the* **theories** *properly called* **particular***. (ibid., p. 229)*

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- **Multiple structures**: suitable combinations of more than one of the above (e.g, *algebraic topology*)
- **Particular structures**: substructures of mother- and multiple structures (e.g., *number theory*)

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  These axioms have different realisations (*groups*, *real numbers*, *displacements*, cf. [Bourbaki, 1950], p. 224).
- Transition from a **mother**-structure to a **multiple** and, finally, a **particular** structure through *adding* new axioms.

From Mathematics to Set Theory

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Task 2

Elaborate on the relationship between new form of STS and a new 'architecture of mathematics'.

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Example (Pluralism)

CH is *neither* true nor false. There is a universe where CH is *true*, and one where it is *false*.

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- and so on..

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Question

Is there a *principled* way to express the transition from a **mother-set-theoretic** structure (and **theory**) to a **particular** structure?

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Question 2

Do we currently have a theory of sets that is able to accommodate the rough sketch above? Answer: MAYBE.

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- it is, crucially, based on **ZFC and Large Cardinal Axioms**
- it may have a **core** (a 'minimal' *definable* world included in all other worlds)

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- Equivalent to (intuitively motivated) reflection principles (e.g., [Bagaria, 2023])

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Theorem (Bagaria, T.)

If there is an extendible cardinal, then the multiverse of the theory MV_T, where T =ZFC+'there is a proper class of extendible cardinals' has a core. Moreover, the existence of the core is consistent with many mutually incompatible extensions of ZFC.

Steel's Multiverse: Motivation/Cont'd.

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Desideratum 1. The **mother-theory** should be ZFC+LCA (**mother-structure** = the **core** of MV)

Desideratum 2. The **particular structures** should be **forcing extensions** of (forcing extensions of.. and so on) of the core

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- Steel's *MV* theory suggests the existence of a unitary 'architecture of set theory' based on large cardinals and forcing, consisting of structures that can play the role of *mother*- and *particular* structures
- *MV*'s architecture helps overcome the 'universe/multiverse debate'
- Steel's *MV*, interpreted against the theoretical background of BS, helps define a version of STS also encompassing set theory

Main issues

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1. (assuming 'architecture of set theory' = 'architecture of maths') Anti-reductionism: maths $\neq$ set theory
2. Multiverse structuralism might not be able to elucidate the notion of '(set-theoretic) structure'
3. Why are LCAs part of the mother-theory?
4. What is a 'natural' theory? Concept is vague (also see [Bagaria and Ternullo, 2021])

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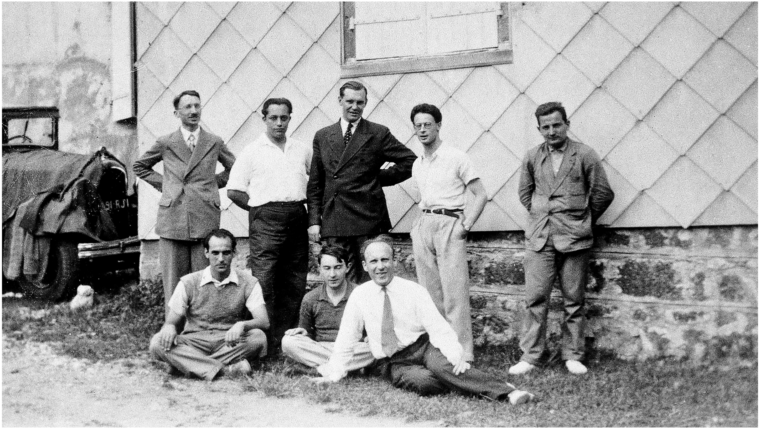

Thanks for your attention!

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