Groups and Fields in Higher Classification Theory

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  $\mathbb{R} \models \exists x : x^2 = (1+1)$ and $\mathbb{Q} \not\models \exists x : x^2 = (1+1)$

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Let $\mathcal{L}$ be a language and $(M, \ldots)$ and $(N, \ldots)$ be any two $\mathcal{L}$–structures.

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Model theorist do not want to distinguish between elementary equivalent structures. We are interested in understanding rather the theory of a structure than the particular structure itself. Hence we study all **models** of a given theory or any theory with a particular property.

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$\Rightarrow$ Examine the definable sets of a structure, i.e. set that can be expressed by a first order formula.

For a $\mathcal{L}$–structure M, a parameter set $A \subset M$ and an $\mathcal{L}(A)$–formula $\phi(x)$ (where x is an n–tuple), we write $\phi(M)$ for the set of realizations of ϕ in M, i.e.

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The notion of definable sets generalizes for example the concept of algebraic varieties and constructible sets in algebraic geometry. For instance, Chevalley's theorem shows that in an algebraically closed field, definable sets are constructible sets.

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Definable sets in $\mathbb{R}^1$.

- $(\mathbb{R})$ as an infinite sets: The only sets one can define are given by finite conjunctions of equalities and inequalities, i.e. finite and cofinite sets. Such structures are called **strongly minimal**.

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- In $(\mathbb{R}, +, \cdot, 0, 1)$ viewed as field. One can define the order as follows:

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In higher dimension, in $(\mathbb{R}, +, \cdot, 0, 1)$ one can define the zero sets of polynomial in multiple variables, i.e. $\mathbb{R}$–rational points of an affine varieties.

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- The intersection of **finitely** many definable subgroups is definable.
- Given a definable group H, we can define
  $C_G(H) : \phi(x, \bar{a}) \equiv \forall y \in H\ xy = yx.$
  $N_G(H)$: $\phi(x, \bar{a}) \equiv \forall y \in H \rightarrow x^{-1}yx \in H;$

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They are the **infinite** union of definable sets but **not** necessarily definable themselves.

1. Introduction
   - Notations and Examples
   - Classification Theory
2. n–dependent theories
3. Fields

Back to tame and wild structures

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$\mathcal{T}$ is **stable**, if R does not encode a linear order.

Coding a Linear Order

Assume we have two sequences $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ as below and a formula $\phi(x, y)$.

$$a_0 \quad a_1 \quad a_2 \quad \dots \quad a_{n-1} \quad a_n \quad a_{n+1} \quad \dots$$

$$b_0 \quad b_1 \quad b_2 \quad b_3 \quad \dots \quad b_n \quad b_{n+1} \quad b_{n+2} \quad \dots$$

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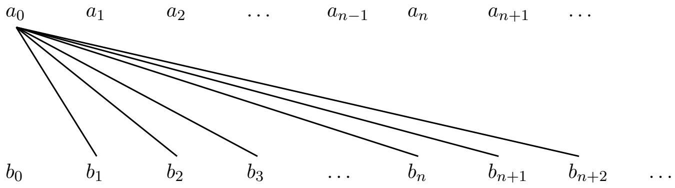

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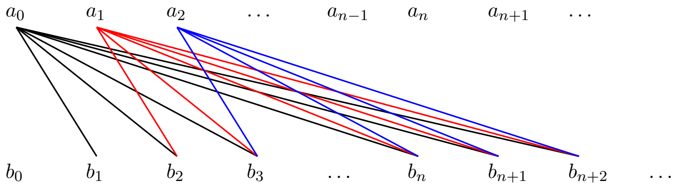

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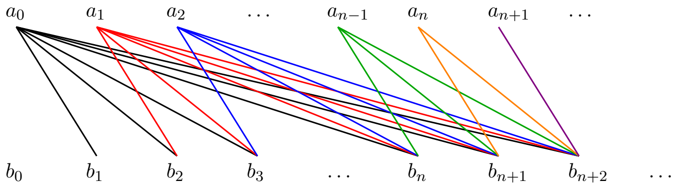
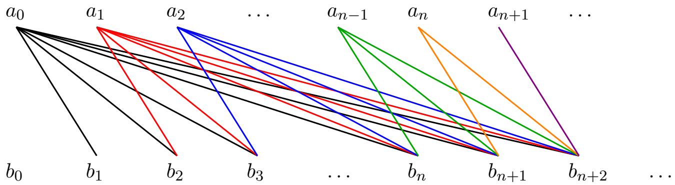


This defines linear order on $(a_i, b_i)_{i\in\omega}$ by

$$(a_i, b_i) < (a_j, b_j) \iff \models \phi(a_i, b_j)$$

Stable Theories: Formal definition



Definition

A formula $\phi(x, y)$ has the **order property** *if there are a sequences of tuples $(a_i : i \in \omega)$ and $(b_j : j \in \omega)$ in $\mathcal{M}$ such that*

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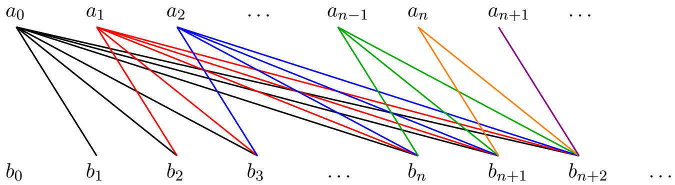

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A theory is called **stable**, *if no formula has the order property.*

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In $(\mathbb{R}, +, \cdot, 0, 1)$, we can define the order $<$ by

$$a < b \iff \exists y : y \neq 0 \wedge a + y^2 = b$$

Hence this field is unstable. However the definable relations are still well–behaved. We will see later more on this.

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For example there exists a notion of independence (so called **forking independence** $\downarrow$) that coincide with algebraic independence in $\mathbb{C}$ and linear independence in vector spaces and thus generalizes these notion of independence to all stable theories.

Stepping outside of stable theories

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Moreover, some other fundamental mathematical object, such as the reals and other o–minimal structures, and valued fields such as the p–adics fall into another class of theories: the NIP theories.

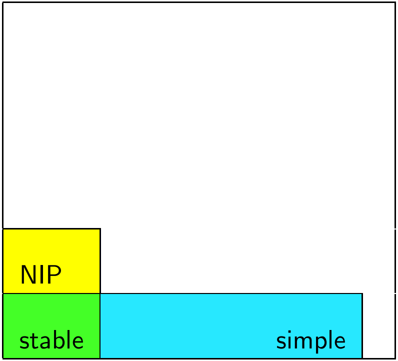
These two classes of theories, also introduced by Shelah, generalize stable theories in two complementary directions. They often still allow key stability–theoretic concepts of independence and dimension to be applied to mathematically important examples which are far from stable.

The Map

$$\text{simple} + \text{NIP} \quad \Rightarrow \quad \text{stable}$$

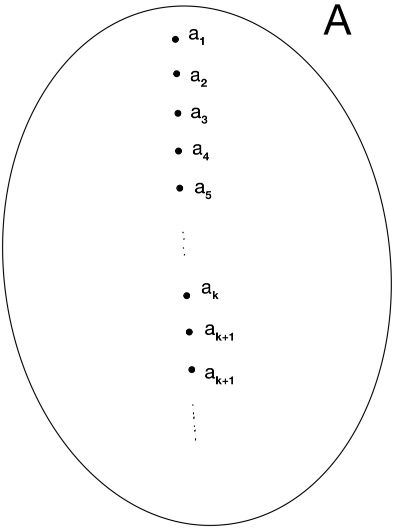
The Map

simple + NIP $\Rightarrow$ stable



NIP theories

Given $\phi(x,y)$ and $(a_i)_{i \in I}$



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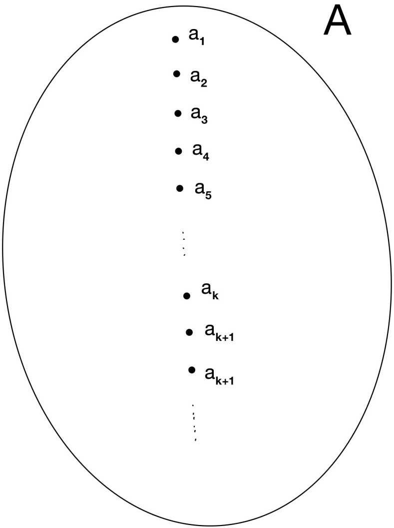

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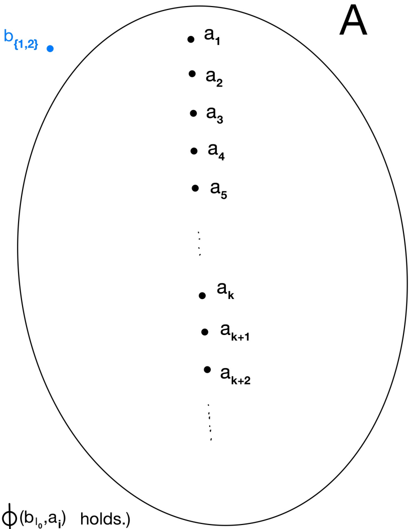

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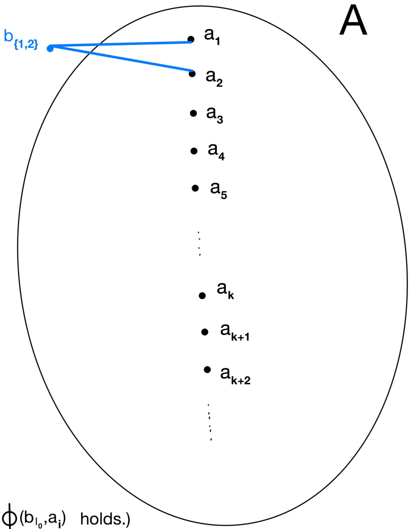

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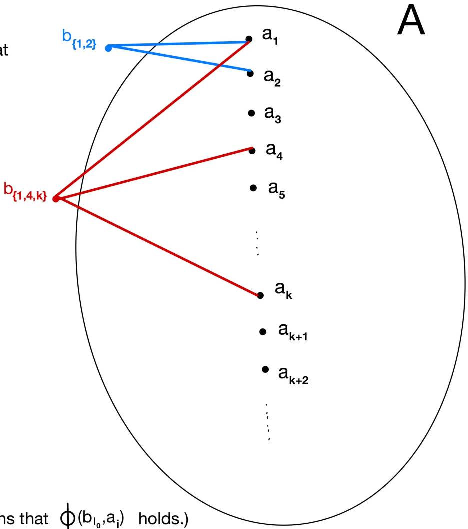

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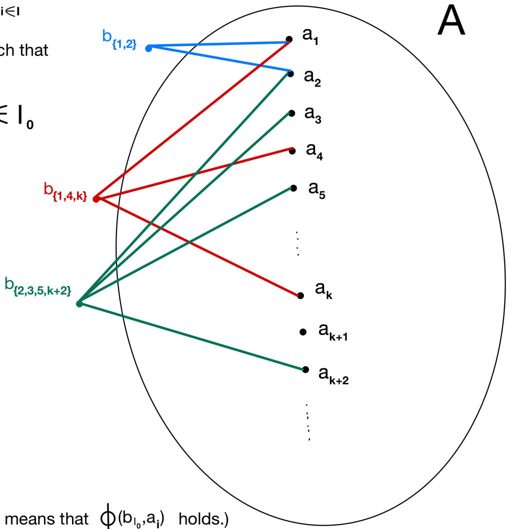

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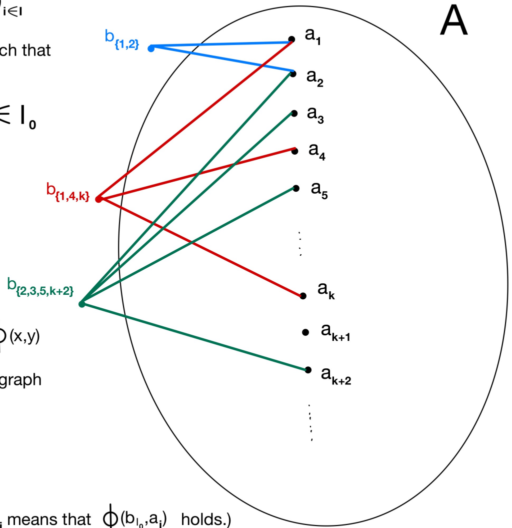

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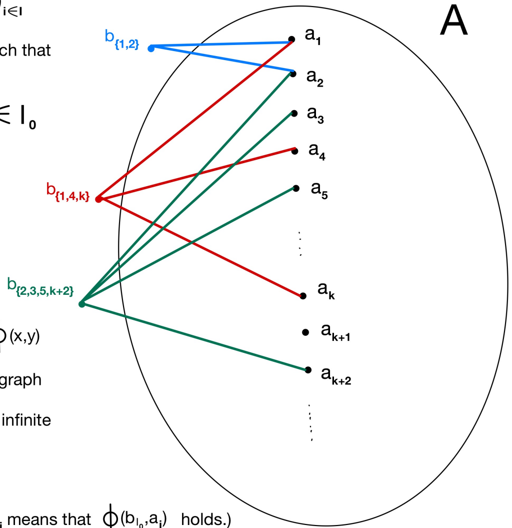

Examples

The Relation $R(x,y) \iff \phi(x,y)$

- does not encode the random graph
- cannot define all subset of an infinite set (here the set $\{a_i : i \in I\}$)

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NIP theories: Formal Definition

Let $\mathcal{T}$ be a $\mathcal{L}$–theory, $\mathcal{M} = (M, \dots)$ be a model of $\mathcal{T}$, $\phi(x, y)$ be a $\mathcal{L}(A)$–formula for some $A \subset M$.

The formula $\phi(x, y)$ has the **independence property** (referred to as **IP**) if there are tuples $(a_i : i \in \omega)$ and $(b_I : I \subseteq \omega)$ in $\mathcal{M}$ such that

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Examples

- stables theories
- $(\mathbb{R}, +, \cdot, 0, 1)$, in general any real closed fields
- algebraically closed valued fields
- ordered abelian groups

Non–Example: Bilinear forms on Vektor spaces

Let $G = \bigoplus_{i \in \omega} \mathbb{F}_p$ and $\mathcal{G} = (G, \mathbb{F}_p, +_G, \cdot)$, where $\bar{a} \cdot \bar{b} = \sum_{i \in \omega} a_i b_i$

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Then, using compactness, we obtain that $x \cdot y = 1$ witnesses *IP*

Connected Components: A Tool in NIP Theories

Let A be a small parameter set. We define:

$$G_A^0 = \bigcap\{H \leq G : H \text{ is } A\text{–definable of finite index}\}$$

$$G_A^{00} = \bigcap\{H \leq G : H \text{ is } A\text{–type–def. of bounded index}\}$$

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If $G_\emptyset^0 = G_A^0$ (resp. $G_\emptyset^{00} = G_A^{00}$ or $G_\emptyset^{\infty} = G_A^{\infty}$) we say that the **definable/type–definable/invariant connected component exist**.

Theorem (Shelah, Gismatullin)

All three connected components exist for NIP group.

n–dependent theories

Motivation

Neostability: Stable, simple, NIP, NTP_2, NTP_1, ...

study definable binary relations $R(x, y)$.

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Example (Stable: Stationary of forking)

Let $\mathcal{T}$ be a stable theory and $p(x)$ and $q(x)$ be types over $M \models \mathcal{T}$. Then there is a unique type $r(x, y)/M$ such that

$$(a, b) \models r \quad \iff \quad a \models p,\ b \models q \text{ and } a \mathop{\smile\!\!\!\!\!\!\mid}_M b$$

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They are many attempts to generalize the binary versions (stable, NIP, simple, etc.) to higher arities.

The most prominent and most studied one is the hierarchy of *n*–**dependent theories**, a higher arity version of NIP theories. We will concentrate the rest of the talk on these kind of theories.

2–dependent Theories

Given $\phi(x,y,z)$ and $(a_i)_{i\in I}$ $(b_i)_{i\in I}$

AxB

• $a_1 b_1$	• $a_1 b_2$	• $a_1 b_3$	• $a_1 b_k$
• $a_2 b_1$	• $a_2 b_2$	• $a_2 b_3$	• $a_2 b_k$
• $a_3 b_1$	• $a_3 b_2$	• $a_3 b_3$	• $a_3 b_k$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
• $a_k b_1$	• $a_k b_2$	• $a_k b_3$	• $a_k b_k$
• $a_{k+1} b_1$	• $a_{k+1} b_2$	• $a_{k+1} b_3$	• $a_{k+1} b_k$
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Given $\phi(x,y,z)$ and $(a_i)_{i\in I}$ $(b_i)_{i\in I}$

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⋮	⋮	⋮	⋮
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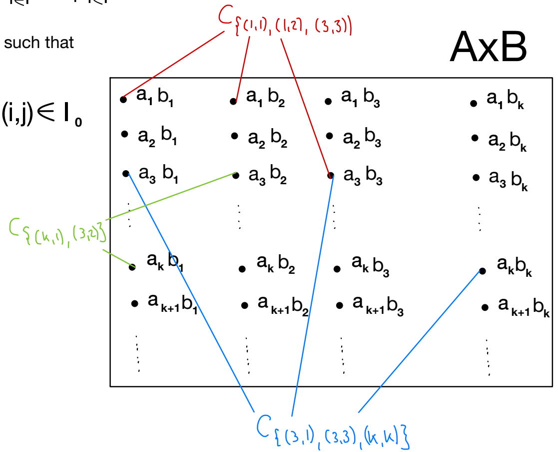

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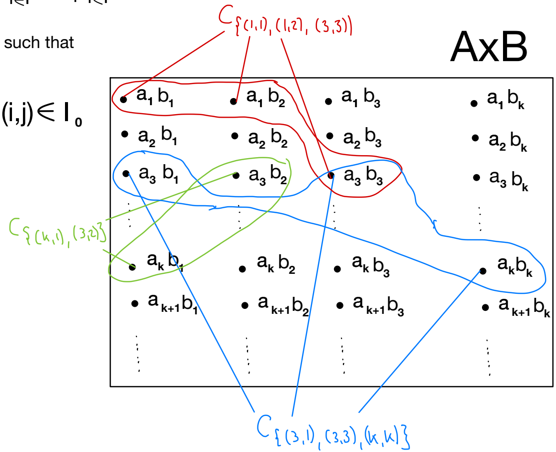

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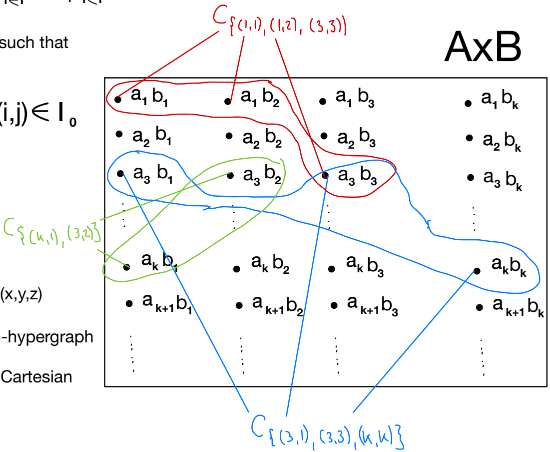

The Relation $R(x,y,z) \iff \phi(x,y,z)$

- does not encode the random n-hypergraph
- cannot define all subset of the Cartesian product of two infinite set

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n–dependent Theories: Formal Defintion

There is no formula $\phi(\bar{y}_0, \ldots, \bar{y}_{n-1}; \bar{x})$ and tuples $(\bar{a}_i^j : i \in \omega, j \in n)$ and $(\bar{b}_I : I \subset \omega^n)$ in $\mathcal{M}$ such that

$$\mathcal{M} \models \psi(\bar{a}_{i_0}^0, \ldots, \bar{a}_{i_{n-1}}^{n-1}, \bar{b}_I) \text{ if and only if } (i_0, \ldots, i_{n-1}) \in I.$$

For any natural number n, a structure is **n–dependent** if one **cannot** define all subsets of an the cartesian product of n infinite sets, i. e. of $A_1 \times A_2 \times \cdots \times A_n$.

Another way of thinking of these structures, is that there are no definable $(n+1)$–ary relations which are "random".

The Hierarchy

The Hierarchy

2–dep

1–dep = NIP

stable

The Hierarchy

3-dep
2-dep
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⋮
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Examples:

stable: algebraically, separable and differential closed fields, free groups, abelian groups, vector spaces, planar graphs

The Hierarchy

⋮	Examples:
k–dep	
⋮	
3–dep	
2–dep	
1–dep = NIP	$\mathbb{R}$, $\mathbb{Q}_p$, algebraically closed valued fields, ordered abelian groups
stable	algebraically, separable and differential closed fields, free groups, abelian groups, vector spaces, planar graphs

The Hierarchy

⋮	Examples:
k–dep	
⋮	
3–dep	
2–dep	random graph, triangle free random graph
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	Examples:
⋮	
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⋮	
3–dep	random 3-hypergraph
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⋮	
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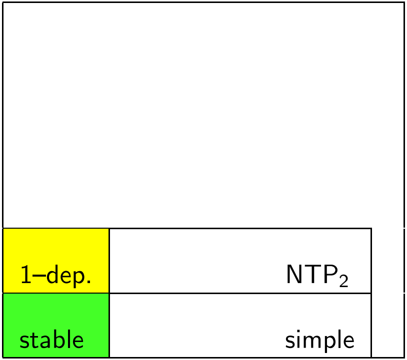

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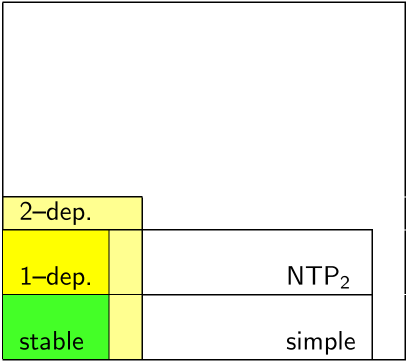

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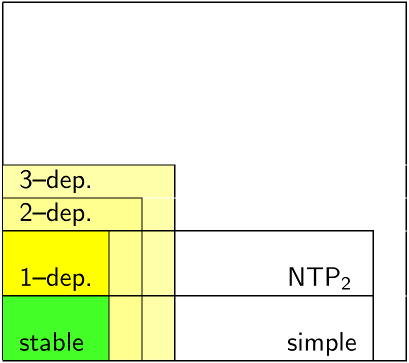

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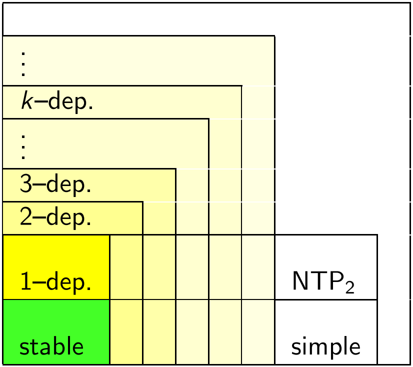

Groups

Using a construction by Mekler we obtained:

Theorem (V./Chernikov, 2019)

For every natural number n there are strictly $n+1$–dependent groups, i.e. groups which are $n+1$ dependent but not n–dependent.

What other algebraic examples exists?

Bilinear forms over arbitrary fields (Grangers Example)

Consider the following 2–sorted structure:

$$\mathcal{M} = (V, K, +_V, \cdot_s, +_K, \cdot_K, \langle ., . \rangle)$$

- V is a vector space over K of infinite dimension with addition $+_V$
- K is a field with with addition $+_K$ and multiplication $\cdot_K$
- $\cdot_s : K \times V \to V$ is scalar multiplication
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symmetric: $\langle x, y \rangle = \langle y, x \rangle$
alternating: $\langle x, x \rangle = 0$
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Reminder

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Question: K NIP $\Rightarrow \mathcal{M}$ is 2–dependent?

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Main obstacle: For $\phi \in \mathcal{L}_K$ the formula

$$\psi(x; y_1, y_2) = \phi(\ \langle x_i^V, x_j^V \rangle\ ,\ \langle x_i^V, y_{lj}^V \rangle\ ,\ \langle y_{lj}^V, y_{mj}^V \rangle\ ,\ x_i^K\ ,\ y_{lj}^K)$$

Compostition Lemma

Theorem

Let $\mathcal{M}$ be an $\mathcal{L}'$–structure such that its reduct to $\mathcal{L} \subset \mathcal{L}'$ is NIP. Let $d \in \omega$, $\phi(x_0, \ldots, x_{d-1}) \in \mathcal{L}$ and y_0, y_1, y_2 be arbitrary variables. For each $0 \leq i < d$, fix some $0 \leq s_i, t_i \leq 2$ and let $f_i : M^{|y_{s_i}|} \times M^{|y_{t_i}|} \to M^{|x_i|}$ be an arbitray binary function. Then the formula

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Then

$$\psi(y_0; y_1, y_2) := \phi(f_1(y_*, y_*), \ldots, f_d(y_*, y_*))$$

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Granger example and n–dependent theories

Reminder

$$\mathcal{M} = (\underbrace{V}_{\textit{inf. dim.}}, K, +_V, \cdot_s, +_K, \cdot_K, \langle .,. \rangle)$$

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Theorem (V., Chernikov, 2021)

If K is NIP then $\mathcal{T}(\mathcal{M})$ is a strictly 2–dependent.

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What about n–linear forms?

n–linear spaces an beyond

Condsider the following 2–sorted structure:

$$\mathcal{M}_n = (V, K, +_V, \cdot_s, +_K, \cdot_K, \langle\, , \ldots , \rangle_n)$$

where we replace the bilinear form by an n–linear form.

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We were able to find a definition of non–degenerate alternating n–linear forms, generalize the Composition Lemma to functions of arity n and show the following:

Theorem (Chernikov, V.)

- *If K is NIP, then $\mathcal{T}(\mathcal{M}_n)$ is strictly n–dependent.*
- *If K has IP_l, then $\mathcal{T}(\mathcal{M}_n)$ has IP_{nl}*

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Moreover, the composition lemma has already been used to show n–dependence of other examples:

Theorem (D'Elbée, Müller, Ramsey, Siniora)

Generic n–nilpotent Lie algebra over $\mathbb{F}_p$ are n–dependent.

Fields

Algebraically and Separably closed fields

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A field K is called **separably closed** if for any separable polynomial (no repeated roots) has a root in K.

Stable Fields

Fact

Algebraically closed and separably closed fields are stable.

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Conjecture

Any stable field is separably closed.

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Theorem (Poizat)

An infinite bounded stable field is separably closed.

Artin Schreier extensions

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A field K of positiv characteristic p is called Artin–Schreier closed if any $a \in K$ the polynomial $x^p - x + a$ has a solution in K.

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Theorem (Kaplan, Scanlon, Wagner)

An infinite NIP field is Artin–Schreier closed.

Valued Fields

Let Γ be an ordered abelian group. Then a **valuation** of a of field K is any map $v : K \to \Gamma \cup \{\infty\}$ which satisfies the following properties for all $a, b \in K$:

- $v(a) = \infty$ if and only if $a = 0$,
- $v(ab) = v(a) + v(b)$,
- $v(a + b) \geq \min(v(a), v(b))$, with equality if $v(a) \neq v(b)$.

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A valuation v is **trivial** if $v(a) = 0$ for all $a \in K \setminus \{0\}$, otherwise it is non–trivial.

Examples

- Let K be any field. The map $v : K \to \{0, \infty\}$ with $v(0) = \infty$ und $v(x) = 0$ for all $x \neq 0$ is a valuation. It is called the **trivial valuation**.
- If $K = \mathbb{Q}$ and p is a prime number, we can write any $x \in \mathbb{Q}^\times$ in a unique way as $p^\nu \frac{c}{d}$ with $c \in \mathbb{Z}$, $d \in \mathbb{N}$ and $\gcd(c, d) = 1$ such that $p \not| c, d$. Setting $v_p(x) = \nu$ gives a valuation on $\mathbb{Q}$ with value group $\mathbb{Z}$. This is called the **p–adic valuation**.
  Examples: $v_p(1) = 0, \quad v_p(p) = 1, \quad v_p(\frac{1}{p^k}) = -k$
- If $K = k(t)$ for some field k and $p \in k(t)$ is irreducible, we can do the same: write $f \in k(t)$ as $p^\nu \frac{g}{h}$ with $g, h \in k(t)$ and $\gcd(g, h) = \gcd(p, g) = \gcd(p, h) = 1$ and set $v_p(x) = \nu$. This is again called the **p–adic valuation** and the value group is again $\mathbb{Z}$.
- If $K = k(t)$, we also have another valuation with value group $\mathbb{Z}$, namely the **degree valuation** v_∞. Here, for $f, g \in k[t] \setminus \{0\}$, we set $v_\infty(\frac{f}{g}) = \deg(g) - \deg(f)$.

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Example

- The trivial valuation.
- $\mathbb{Q}$ together with the p–adic valuation is not henselian.
- However, one can complete $\mathbb{Q}$ with respect to the p–adic absolut value (i.e. $|x|_v = e^{-v(x)}$) and obtains the p–adics numbers $\mathbb{Q}_p$. These are henselian.

Stable Valued Fields

Theorem (Jahnke)

If K is an infinite stable field and v is a non–trivial henselian valuation on K, then K is separably closed.

Towards the Classification of 1–dependent Fields

The two main conjectures for 1–dependent fields are

- The *henselianity conjecture*: any 1–dependent valued field is henselian.
- The *Shelah conjecture*: any 1–dependent field K is algebraically closed, real closed, finite, or admits a non–trivial henselian valuation.

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Theorem (Johnson, 2019)

Any infinite NIP valued field of positive characteristic is henselian.

The main ingredients of model theory are the facts, that NIP fields are Artin–Schreier closed and the existence of the connected component.

Summary of the conjectures for fields

Conjecture

Let K be a infinite field. K is

- *stable* $\iff$ *it is separably closed*
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- *n–dependent* $\iff$ *NIP*

Since it is very hard to prove these conjectures, our first aim was to generalize the known results in NIP theories to the *n*–dependent context.

Henselianity Conjecture for n–dependent Fields

Reminder

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Theorem (Chernikov, V. 2021)

Any infinite n–dependent valued field of positive characteristic is henselian.

Thank you!